Front motion in the one-dimensional stochastic Cahn-Hilliard equation

D. C. Antonopoulou and D. Blomker and Georgia D Karali

Original Citation:
Antonopoulou, D. C. and Blomker, D. and Karali, Georgia D
(2011)
Front motion in the one-dimensional stochastic Cahn-Hilliard equation.
(Submitted)
This version is available at: http://preprints.acmac.uoc.gr/69/
Available in ACMAC’s PrePrint Repository: February 2012
ACMAC’s PrePrint Repository aim is to enable open access to the scholarly output of ACMAC.

http://preprints.acmac.uoc.gr/
FRONT MOTION IN THE ONE-DIMENSIONAL STOCHASTIC CAHN-HILLIARD EQUATION

D.C. ANTONOPOULOU†¶, D. BLÖMKER‡, G.D. KARALI†¶

Abstract. In this paper, we consider the one-dimensional Cahn-Hilliard equation perturbed by an additive noise and study the dynamics of interfaces for the new stochastic model. The noise is smooth in space and is defined as a Fourier series with independent Brownian motions in time. Motivated by the work of Bates & Xun on slow manifolds for the integrated Cahn-Hilliard equation, our analysis reveals the significant difficulties and differences in comparison with the deterministic problem. New higher order terms that we estimate appear due to Itô calculus and stochastic integration and dominate the exponentially slow deterministic dynamics. Using a local coordinates system and defining the admissible interfaces positions as a multi-dimensional diffusion process we derive a first order linear system of stochastic ordinary differential equations approximating the equations of front motion. Furthermore, we prove stochastic stability for the approximate slow manifold of solutions on a very long time scale and evaluate the noise effect.

Keywords: 1-D Stochastic Cahn-Hilliard, slow manifold, interface motion, additive noise, dynamics, stability.

1. Introduction

1.1. The problem. The standard Cahn-Hilliard equation is a simple model for the phase separation of a binary alloy at a fixed temperature, proposed in [17, 18]. This model was extended by Cook [23, 40] in order to incorporate thermal fluctuations in the form of an additive noise. In this paper, we consider the one-dimensional Cahn-Hilliard equation posed on (0, 1) with an additive stochastic term:

(SC-H) \[ u_t = (-\varepsilon^2 u_{xx} + f(u))_{xx} + \partial_x \dot{W}, \quad 0 < x < 1, \quad t > 0, \]

with no-flux boundary conditions of Neumann type:

(1.1) \[ u_x = u_{xxx} = 0 \quad \text{at} \quad x = 0, 1. \]

Here, \( \dot{W} \) is a smooth in space space-time noise defined as the formal derivative of a Wiener process \( W \). The nonlinearity \( f = f(u) \) is the derivative of a smooth double equal-well potential \( F \) taking its global minimum value 0 at \( u = \pm 1 \) [1], with non-degenerate minima. A typical example is \( F(u) := \frac{1}{4}(u^2 - 1)^2 \) with \( f(u) := u^3 - u \). The parameter \( \varepsilon > 0 \) is a small atomistic interaction length modeling the width of layers that develop during the initial phase separation of spinodal decomposition (cf. [12, 13]). In the later stages of the separation process \( \varepsilon \) measures the width of interfacial regions between the pure phases \( u = \pm 1 \).

A characteristic feature of the Cahn-Hilliard model is the conservation of total mass \( \int_0^1 u(t, x)dx \), which we now fix to be \( M \in (-1, 1) \). Substituting \( \tilde{u}(t, x) := \int_0^x u(t, y)dy \) we obtain the equivalent integrated stochastic Cahn-Hilliard equation:

(ISC-H) \[ \tilde{u}_t = -\varepsilon^2 \tilde{u}_{xxxx} + (f(\tilde{u}_x))_x + \dot{W}, \quad 0 < x < 1, \quad t > 0, \]

associated with the boundary conditions:

(1.2) \[ \tilde{u}(t, 0) = 0, \quad \tilde{u}(t, 1) = M, \]
\[ \tilde{u}_{xx}(t, 0) = \tilde{u}_{xx}(t, 1) = 0. \]
J. Carr and R. Pego in [21, 22] presented a detailed analysis of the slow evolution of patterns of the singularly perturbed Ginzburg-Landau equation. They proved existence and persistence of metastable patterns and analyzed the equations governing their motion. These metastable states have been characterized in terms of the global unstable manifolds of equilibria. In [7, 8], P.W. Bates and J. Xun extended their argument and studied the dynamics of the one-dimensional C-H equation in a neighborhood of an equilibrium having \( N + 1 \) transition layers, using several estimates presented in [21, 22]. They determined the exponentially slow speed of the layer motion and described precisely the layer motion directions. In addition, they established existence of an \( N \)-dimensional unstable invariant manifold attracting solutions exponentially fast uniformly in \( \varepsilon \). Related works in this direction are [9, 33, 41].

Motivated by the work of Bates and Xun for the deterministic problem, we study dynamics for the stochastic model. Due to stochastic integration, new higher order terms appear that we estimate using techniques and ideas of [7, 8, 21, 22]. In the sequel we shall refer frequently to some important definitions and results presented in the aforementioned articles, therefore, we give some details concerning our notation. Following [21, 22], we use the letter \( f \) for the nonlinearity in (SC-H), and denote by \( F \) the double equal well potential. In [7, 8] the symbol \( W' \) is used in place of \( f \); we avoided such a notation since we name by the standard symbol \( \dot{W} \) the additive noise.

1.2. The effect of noise. The stochastic Cahn-Hilliard equation being one of the important examples of the nonlinear Langevin equations is based on a field-theoretic approach to the non-equilibrium dynamics of metastable states (see for example [23, 37, 40]). The multi-dimensional generalized stochastic Cahn-Hilliard equation associated with Neumann boundary conditions posed on bounded domains contains a time dependent noise into the chemical potential and an additive multiplicative noise defined as the formal derivative of a Wiener process. The chemical potential noise describes external fields [37, 35, 39], while the free-energy independent noise may describe thermal fluctuations or external mass supply [23, 40, 37, 35].

Existence and uniqueness of solution for the stochastic problem was first studied in [24], where the nonlinearity \( f \) is a polynomial of odd degree and the problem is posed on multi-dimensional rectangular domains. Further, in [19], the author proved existence of solution and of its density for the stochastic Cahn-Hilliard with additive noise (in the sense of Walsh, cf. [44]) posed on cubic domains. When the trace of the Wiener process is finite, existence was analyzed in [28]. In [4], existence for the generalized stochastic Cahn-Hilliard equation was derived for general convex or Lipschitz domains; the main novelty of this work was the derivation of space-time Hölder estimates for the Green’s kernel of the stochastic problem, by using the domain’s geometry, which can be very useful in many other circumstances. The polynomial nonlinearity which forces the solution to stay between the pure phases \( \pm 1 \) has been analyzed in [12, 13, 19, 20, 21, 28], while in [27, 26, 31] a stochastic Cahn-Hilliard with reflection was considered.

In [12, 13] (see [14] for a review), the effect of noise on evolving interfaces during the initial stage of phase separation is analyzed. The evolution of these interfaces is stochastic and not yet fully understood. In [12], the authors show that for a solution starting at the homogeneous state, the probability of staying near a certain finite-dimensional space of pattern is high as long the solution stays within the distance of the size of the homogeneous state. Further, in [13], the dynamics of a nonlinear partial differential equation perturbed by additive noise are considered. Under the assumption that the underlying deterministic equation has an unstable equilibrium, the authors show that the nonlinear stochastic partial differential equation exhibits essentially linear dynamics far from equilibrium.

On the other hand interface motion has been studied for many related models like Allen-Cahn or Ginzburg Landau and phase-field models, cf. for example [15, 11] for a rigorous analysis or the results of [30] for formal arguments, which describe the interfaces as interacting Brownian motions. Numerical results for interface motion are presented in [43, 36]. The problem of singular perturbation for a reaction-diffusion stochastic partial differential equation of Ginzburg-Landau type is investigated in [32]. The motion of interfaces for Cahn-Hilliard was only studied in an unpublished note by S. Brassesco in 2003, where she studied a solution with a single interface on \( \mathbb{R} \). When properly rescaled, the interface is driven by non-Markovian dynamics
for the interfaces there exists manifold \( M \) interesting and difficult problem of investigating further the properties of by \( h \) on the whole real-line. The elements of the manifold are parametrized by the position of the fronts given different way, it can be thought of as piecing together a rescaled one kink (or front) of steady state solutions Pego that is based on the analysis of an approximate invariant manifold \( M \), stochastic model in the finite interval \((0, 1)\). The space-time noise that we introduce is smooth in space, so, integration in space is deterministic. Therefore, in order to study the transition layers dynamics for the deterministic Cahn-Hilliard equation and related equations we refer to [6, 16, 17, 29, 31] and the references therein. Results for the noisy Cahn-Hilliard equation are of great interest for the studying of Ostwald ripening [2, 3, 38] and nucleation [10]. For a survey, including numerical results and conjectures concerning the nucleation problem, see [14].

1.3. The approximate slow manifold. The space-time noise that we introduce is smooth in space, so, integration in space is deterministic. Therefore, in order to study the transition layers dynamics for the stochastic model in the finite interval \((0, 1)\), we closely follow the approach of Bates & Xun and Carr & Pego that is based on the analysis of an approximate invariant manifold \( M \). Although constructed in a different way, it can be thought of as piecing together a rescaled one kink (or front) of steady state solutions on the whole real-line. The elements of the manifold are parametrized by the position of the fronts given by \( h \in \mathbb{R}^{N+1} \). Nevertheless, in our case the dependency on time is stochastic. This fact arises the very interesting and difficult problem of investigating further the properties of \( M \) by means of deriving higher order estimates related to the stationary problem.

Let us present first the details necessary for the steady state solutions \( \phi \), the parameters \( h \) and the manifold \( M \). Given \( \varepsilon > 0 \), we consider \( a \) such that \( f'(u) > 0 \) for all \( u \) satisfying \( |u \pm 1| < a \). Then, cf. [21], there exists \( \rho > 0 \) such that if \( \ell \) satisfies \( \varepsilon < \rho \) then a unique solution \( \phi = \phi(x, \ell, \pm 1) \) exists for the following stationary Dirichlet problem

\[
\varepsilon^2 \phi_{xx} - f(\phi) = 0, \quad -\ell/2 < x < \ell/2, \\
\phi = 0, \quad x = \pm \ell/2,
\]

that satisfies: (a) \( \phi(x, \ell, +1) > 0 \) for \( |x| < \ell/2 \), and \( |\phi(0) - 1| < a \), (b) \( \phi(x, \ell, -1) < 0 \) for \( |x| < \ell/2 \), and \( |\phi(0) + 1| < a \). For \( \varepsilon > 0 \) small, it is known that \( \phi = \pm 1 \) with transition layers of order \( \mathcal{O}(\varepsilon) \) near \( x = \pm \ell/2 \).

Following [3], we consider the slowly evolving solutions with \( N + 1 \) layers well separated and bounded away from the boundary \( x = 0, 1 \) and define the set of admissible positions \( h \) of the interfaces

\[
\Omega_\rho := \left\{ h \in \mathbb{R}^{N+1} : 0 < h_1 < \cdots < h_{N+1} < 1, \quad \frac{\varepsilon}{\rho} < h_j - h_{j-1}, \quad j = 1, \ldots, N + 2 \right\},
\]

with \( h_0 := -h_1, h_{N+2} := 2 - h_{N+1} \). These interfaces evolve in time, and we expect them to have a width of order \( \varepsilon \). Thus, the distance is bounded below by \( \varepsilon/\rho \) for some small \( \rho \). Later we fix \( \rho = \varepsilon^\kappa \) for any small \( \kappa > 0 \).

Let \( h \in \Omega_\rho \) be given as above, and denote the mid points between interfaces by \( m_j := \frac{h_{j-1} + h_j}{2} \) for \( j = 1, \ldots, N + 2 \) with \( m_0 = 0 \) and \( m_{N+1} = 1 \). Moreover, we define the function \( u^h : I_j := [m_j, m_{j+1}] \to \mathbb{R} \) for the interfaces \( h \) by

\[
u^h(x) = \left[ 1 - \chi \left( \frac{x-h_j}{\varepsilon} \right) \right] \cdot \phi \left( x - m_j, h_j - h_{j-1}, (-1)^j \right) \\
+ \chi \left( \frac{x-h_j}{\varepsilon} \right) \cdot \phi \left( x - m_{j+1}, h_{j+1} - h_j, (-1)^{j+1} \right),
\]

where \( \chi : \mathbb{R} \to [0, 1] \) is a \( C^\infty \) cut-off function such that \( \chi = 1 \) on \([1, \infty)\) and \( \chi = 0 \) on \((-\infty, -1)\).

**Definition 1.1 (Approximate slow manifold).** The first approximate manifold of the stochastic Cahn-Hilliard solution is defined by

\[
\mathcal{M}_1 := \left\{ u^h : h \in \Omega_\rho \right\}.
\]
Fixing a mass \( M \in (-1, 1) \), we define as the second approximate manifold the submanifold \( M \) of \( M_1 \) where mass conservation holds i.e.

\[
M := \{ u^h \in M_1 : \int_0^1 u^h dx = M \}.
\]

For the integrated equation, we consider the manifold

\[
\tilde{M} := \{ \tilde{u}^h : u^h \in M, \tilde{u}^h(x) = \int_0^x u^h dx \}.
\]

**Figure 1.1.** Gluing together positive and negative solutions of (1.3) to obtain \( u^h \in M \).

Note that \( m_1 = 0, m_{N+2} = 1 \), and \( I_j = [m_j, m_{j+1}] \).

**Remark 1.2.** In view of the initial stochastic equation (SC-H), conservation of mass holds if and only if formally

\[
\int_0^1 \partial_x \tilde{W} dy = \tilde{W}(1) - \tilde{W}(0) = 0.
\]

This is later assured by our assumptions on \( W \), which impose Dirichlet-boundary conditions (cf. Definition 2.2 and Assumption 2.3). A very simple rigorous example is the following: consider \( \tilde{W} := g(x) \tilde{V}(t) \), where \( \tilde{V}(t) \) is a white noise in time and \( g \) a smooth function satisfying \( g(1) = g(0) \), then by integrating in space the equation (SC-H) and using the fact that

\[
\int_0^1 \partial_x \tilde{W} dy = \tilde{V}(t) \int_0^1 g_x(y) dy = 0,
\]

we obtain mass conservation even with the noise. We can extend this example to infinite series of terms of these type.

Throughout the entire paper we will assume that the additive noise in (SC-H) satisfies (1.6), and therefore the proposed stochastic model exhibits mass conservation.

1.4. **The new coordinate system.** Along \( \tilde{M} \) the natural coordinate system would be to use the parameters \( h \in \Omega_\rho \) for the position in \( M \) (where \( N \) of them are sufficient due to mass conservation), together with the orthogonal projection onto \( \tilde{M} \). In order to relate the coordinate system to the deterministic flow of (ISC-H), one approximates the tangential space of \( \tilde{M} \) by the span of some functions \( E_\xi^i, i = 1, \ldots, N \) to be defined in the sequel; here, we follow [7].

We denote the \( L^2(0,1) \) inner product by \( \langle u, v \rangle := \int_0^1 uv dx \), the induced \( L^2 \)-norm by \( \| \cdot \| \) and introduce the symbol \( \tilde{g}(x, t) := \int_0^x g(y, t) dy \), for any \( g \) smooth in space.

Due to mass conservation, we reduce the parameter space \( \Omega_\rho \) by one dimension, define

\[
\xi := (\xi_1, \ldots, \xi_N) = (h_1, \ldots, h_N),
\]
and consider $h_{N+1}$ as a function of $\xi$. Thus, for $\tilde{u}_j^h := \frac{\partial \tilde{u}_j}{\partial h_j}$ and $\tilde{u}_j^\xi := \frac{\partial \tilde{u}_j}{\partial \xi_j}$, we obtain that

$$\tilde{u}_j^\xi = \frac{\partial \tilde{u}_j^h}{\partial h_{N+1}} \cdot \frac{\partial h_{N+1}}{\partial h_j} + \frac{\partial \tilde{u}_j^h}{\partial h_j}.$$

We use the following coordinate system around $\tilde{M}$: $\tilde{u} \rightarrow (\xi, \tilde{v})$, where we write the stochastic solution $\tilde{u}$ of (ISC-H) as a sum of stochastic processes

$$\tilde{u}(t) := \tilde{u}_\xi(t) + \tilde{v}(t).$$

Here the position on $\tilde{M}$ is given by $\tilde{u}_\xi \in \tilde{M}$ while the distance from $\tilde{M}$ is given by $\tilde{v}$ which is defined as the following projection such that

$$\langle \tilde{v}, E_\xi^j \rangle = 0 \quad \text{for} \quad j = 1, \ldots, N.$$

It turns out that the functions $E_\xi^j$ are good approximations to the first eigenfunctions of the linearized integrated Cahn-Hilliard operator, which in turn are good approximations to the tangential space of $\tilde{M}$. They are defined as follows:

$$E_\xi^j := \tilde{w}_j(x) - Q_j(x), \quad \tilde{w}_j := \tilde{u}_j^h(x) + \tilde{u}^\xi_{j+1}(x),$$

$$Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\tilde{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\tilde{w}_{jxx}(1) + x\tilde{w}_j(1), \quad j = 1, \ldots, N,$$

where the $Q_j$ later turn out to be exponentially small terms (cf. [7]), that only takes care of the boundary values of $E_\xi^j$.

For short-hand notation, we also define higher derivatives using indices

$$E_{\xi i}^k := \frac{\partial E_\xi^i}{\partial \xi_k}, \quad E_{\xi i k}^\xi := \frac{\partial^2 E_\xi^i}{\partial \xi_i \partial \xi_k}, \quad \tilde{u}_{\xi i k} := \frac{\partial^2 \tilde{u}_\xi}{\partial \xi_i \partial \xi_k}.$$

The rest of the paper is organized as follows: In Section 2, we present the main results including a proper definition of the noise, the equations for the motion of the interfaces, the stability of the approximate manifold $\tilde{M}$, and the approximation of the front motion in a neighborhood of $\tilde{M}$. The proofs of the main results appear in Sections 3 and 4, while the final Section 5 collects all the estimates of the higher order terms appearing in the stochastic dynamics. Here, we consider the stationary problem (1.3) and analyze the properties of its solution by deriving bounds for higher order derivatives, extending some of the results of [21, 22, 7, 8].

2. Main results

The SDE (Stochastic Differential Equation) system for the motion of fronts is given by the projection onto the manifold $\tilde{M}$, using the coordinate system of Section 1.4. We then prove that $\tilde{M}$ is locally exponentially attracting and show that solutions stay with high probability in a small slow tube around $\tilde{M}$, until large
times or until one of the layers becomes small. The flow along \( \mathcal{M} \) is well described by the SDE for the interfaces \( \xi \). Depending on the strength of the noise we investigate how the equation of motion of the fronts looks like and evaluate the noise effect. In addition, we investigate extensively the case \( N = 1 \) where the motion of the second interface is determined by the first. Finally, the case of space-time white noise is discussed. In the last section, we present the proofs of the estimates used in our analysis concerning all the higher order terms that appear in the stochastic setting. These are technical results that are independent of the other sections.

Let us first explain briefly how the equations of motions along \( \mathcal{M} \) are derived in Section 3; for details we refer to Subsection 3.2. If \( \tilde{u} \) is the solution of (ISC-H), then using the Itô-formula to differentiate \( \tilde{u}^\xi \) in \( t \) we get

\[
\begin{align*}
\langle \tilde{u}^\xi, d\tilde{u} \rangle &= \sum_{j=1}^N \langle \tilde{v}^\xi_j, d\xi_j \rangle + \frac{1}{2} \sum_{1 \leq k, l \leq N} \langle \tilde{u}^\xi_k d\xi_k d\xi_l + d\mathcal{V} \rangle.
\end{align*}
\]

We take the inner product in space of (ISC-H) with \( \langle \tilde{u}^\xi, \tilde{v}^\xi \rangle \) to obtain for any \( i = 1, \ldots, N \):

\[
\begin{align*}
\langle \tilde{E}^\xi_i, d\tilde{u} \rangle &= \langle -\varepsilon^2 \tilde{u}_{xxxx}^\xi + (f(\tilde{u}_x^\xi))_x, \tilde{E}^\xi_i \rangle dt + \langle \tilde{E}^\xi_i, dW \rangle.
\end{align*}
\]

The inner product of (2.1) with \( \tilde{E}^\xi_i \) now gives

\[
\begin{align*}
\langle \tilde{E}^\xi_i, d\tilde{u} \rangle &= \sum_{j=1}^N \langle \tilde{u}^\xi_j, \tilde{E}^\xi_i \rangle d\xi_j + \frac{1}{2} \sum_{1 \leq k, l \leq N} \langle \tilde{u}^\xi_k d\xi_k, \tilde{E}^\xi_l d\xi_l + \langle \tilde{E}^\xi_i, d\mathcal{V} \rangle \rangle.
\end{align*}
\]

Applying the Itô-formula in differentiating in \( t \) the term \( \langle \tilde{v}, \tilde{E}^\xi \rangle = 0 \), using \( d\tilde{u} = d\tilde{u}^\xi + d\mathcal{V} \) and combining (2.2) with (2.3), we get for \( i = 1, \ldots, N \) the following system in \( d\xi_1, \ldots, d\xi_N \) for the stochastic Cahn-Hilliard equation:

\[
\begin{align*}
\sum_j \left[ \langle \tilde{v}^\xi_j, \tilde{E}^\xi_i \rangle - \frac{1}{2} \langle \tilde{u}_{xxxx}^\xi + \tilde{v}_{xxxx}, \tilde{E}^\xi_i \rangle \right] d\xi_j = & \langle -\varepsilon^2 \tilde{u}_{xxxx}^\xi + \tilde{v}_{xxxx} + (f(\tilde{u}_x^\xi + \tilde{v}_x^\xi))_x, \tilde{E}^\xi_i \rangle dt \\
& + \sum_{l,k} \left[ \frac{1}{2} \langle \tilde{u}^\xi_l, \tilde{E}^\xi_k \rangle - \frac{1}{2} \langle \tilde{u}^\xi_k, \tilde{E}^\xi_l \rangle - \langle \tilde{u}^\xi_k, \tilde{E}^\xi_l \rangle \right] d\xi_l d\xi_k \\
& + \sum_j \langle dW, \tilde{E}^\xi_{ij} \rangle d\xi_j \\
& + \langle \tilde{E}^\xi_i, dW \rangle.
\end{align*}
\]

In the above, we denote that the last three additive terms at the right-hand side give the difference from the deterministic Cahn-Hilliard system of [8].

**Remark 2.1.** In view of (2.4), we observe that the study of dynamics for the stochastic Cahn-Hilliard, even in one dimension, arises a much more complicated and difficult problem in comparison with the deterministic one.

1. **Deterministic case:** The system is linear in \( d\xi_j \), therefore by estimating the inverse matrix on the left-hand side (possibly close to \( \mathcal{M} \)) and the right-hand side terms, the motion of interfaces is obtained, see [8].

2. **Stochastic case:** Obviously, for a general noise definition the system is non-linear due to the appearance of \( d\xi_l d\xi_k \). In the sequel, we make an ansatz for \( \xi \) in order to get a linear system, which then justifies the ansatz. Further, we need estimates for the additional higher order terms \( \tilde{E}^\xi_{ij}, \tilde{E}^\xi_{ikl}, \) and \( \tilde{u}^\xi_{kl} \). Here we need to improve the estimates of [7].

The sufficiently regular noise \( \tilde{W} \) is the formal derivative of a Wiener process \( W \) defined as follows.
**Definition 2.2 (The Wiener process W).** Let $W$ be a $\mathcal{Q}$-Wiener process in the underlying Hilbert-space $H = L^2(0, 1)$, $\mathcal{Q}$ a symmetric operator and $(\epsilon_k)_{k \in \mathbb{N}}$ an orthonormal basis with corresponding eigenvalues $\alpha_k^2$, such that

$$\mathcal{Q}\epsilon_k = \alpha_k^2 \epsilon_k \quad \text{and} \quad W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k(t) \epsilon_k,$$

for a sequence of independent real-valued standard Brownian motions $\{\beta_k(t)\}_{t \geq 0}$ (cf. DaPrato, Zabzcyck [20]).

We will always use the following assumption, which is an assumption for mass conservation and regularity.

**Assumption 2.3.** Suppose that the $\epsilon_k$ are also the eigenfunctions of the Dirichlet-Laplacian. Moreover, we assume that for some $0 < \delta_\varepsilon$

1. $\|\mathcal{Q}\| < C\delta_\varepsilon^2$,
2. $\sum_{k=1}^{\infty} \alpha_k^2 B_\varepsilon(\epsilon_k) < C\delta_\varepsilon^2$,
3. $\|\partial_x \mathcal{Q}\| < C\delta_\varepsilon^2$.

where for some small $\kappa > 0$ we have $\delta_\varepsilon < \varepsilon^{9/(2-\kappa)}$.

The first assumption on the norm of $\mathcal{Q}$ as an operator in $H$ means that the strength of the noise is bounded by $O(\delta_\varepsilon)$, while the second and third one are additional assumptions on the noise regularity. Note that

$$B_\varepsilon(\epsilon) = \varepsilon^2 \|e_{xx}\|^2 + \|e_x\|^2,$$

which is equivalent to the standard $H^2$-norm (see (3.14)).

The next crucial assumption considered in order to obtain the equation for the interfaces $\xi$ is the following. Let $\tilde{u}$ be a solution of (ISC-H), then let $\xi(t)$ be a diffusion process in $\mathbb{R}^N$ defined for any $k = 1, \ldots, N$ by

$$d\xi_k = b_k(\xi)dt + \langle \sigma_k(\xi), dW \rangle,$$

for some vector field $b : \mathbb{R}^N \to \mathbb{R}^N$ and some variance $\sigma : \mathbb{R}^N \to H^N$.

We define as in [8] the matrix

$$A_{ij}(\xi) = \langle \tilde{u}_{\xi}^j, E_{\xi}^i \rangle - \langle \tilde{v}, E_{ij}^\xi \rangle,$$

which is invertible, provided that we are near the slow manifold. The previous noise definition combined with (2.4), gives the following SDE system for the interfaces motion for the stochastic C-H:

$$\sum_j A_{ij}(\xi)d\xi_j = -(\varepsilon^\xi(\tilde{u}_{\xi,xxx}^j + \tilde{v}_{xxx}) + (f(\tilde{u}_{\xi}^j + \tilde{v}_x)_{xx}, E_{\xi}^i))dt$$

$$+ \sum_{l,k} \left[ \frac{1}{2} \langle \tilde{v}, E_{\xi}^l \rangle - \frac{1}{2} \langle \tilde{u}_{\xi}^l, E_{\xi}^j \rangle - \langle \tilde{u}_{\xi}^l, E_{\xi}^j \rangle \right] \langle Q\sigma_k(\xi), \sigma_l(\xi) \rangle dt$$

$$+ \sum_j \langle QE_{\xi}^j, \sigma_j(\xi) \rangle dt$$

$$+ \langle E_{\xi}^j, dW \rangle.$$

(cf. also the equivalent presentation (3.11)). We can easily read off $b$ and $\sigma$ from this equation for $\xi$. Moreover, this equation gives the flow along $\mathcal{M}$ by describing the interface positions. It is now easy to check by construction that the difference $\tilde{v} = \tilde{u} - \tilde{u}^\xi$ is actually the $\tilde{v}$ of the coordinate system (see Sec. 1.4). In addition, a solution of (2.5) together with a corresponding equation for $\tilde{v}$ (see (3.16), later) describes a solution $\tilde{u}$ of (ISC-H).

Further, in Section 3 the variance $\sigma$ of the multi-dimensional diffusion process $\xi$ of the interfaces is computed first explicitly and then estimated in terms of $\varepsilon$. A main result of great importance is the stochastic analysis of the stability of the second approximate manifold which is presented in Theorem 3.6 of this section. Over a long time-scale of order $O(\varepsilon^{-q})$ for any $q > 0$, we show that with high probability
the solution of the stochastic Cahn-Hilliard stays in a small neighborhood $\Gamma$ of the integrated manifold $\tilde{M}$, unless an interface breaks down.

In Section 4, we present first Theorem 4.1 in which we approximate (2.5) and derive the equations of interfaces motion. Further, we consider several examples where Theorem 4.1 is simplified. If the noise is exponentially small, then we recover the slow motion results of [7, 8]. There is a slow channel as a neighborhood of $\tilde{M}$, in which with high probability the motion of the interfaces is described by the deterministic regime.

There is also an interesting intermediate regime of still exponentially small noise, which for simplicity of presentation we do not consider in this article. Here, due to the presence of noise, additional deterministic and stochastic terms appear in the deterministic equation of Bates & Xun [8]. An interesting case from the point of applications is the case where the noise strength is a power of $\varepsilon$. As the general case is quite involved in presentation, we consider only two interfaces (i.e. $N = 1$). Here, obviously the motion of the second interface is determined by the first which is approximated by the following SDE (cf. (4.10)):

$$d\xi_1 = \frac{1}{32\ell_2^2} \frac{\partial}{\partial \xi_1} \|Q^{1/2}E_1\|^2 dt + \frac{1}{4\ell_2} \langle E_1^\xi, dW \rangle,$$

where $\ell_2$ is the distance between the two interfaces. Finally in this section, we also discuss the case of non-smooth in space space-time white noise ($Q = Id$), which we do not treat by our assumptions; here $\xi_1$ would be close to a Brownian motion with variance $\delta^2/4\ell_2$.

Section 5 provides estimates for the second order derivatives $\frac{\partial^2 h_{k+1}}{\partial h_{k+1}}, \frac{\partial^2 h_{k+1}}{\partial h_{k+1}^x}$, for the higher order derivatives of $E_1^\xi$ and $\tilde{u}^\xi$, and a bound for the quantity $\langle L^c \tilde{v}, \tilde{u}^\xi_{kl} \rangle$ (needed in the proof of the stability Theorem). Here the operator $L^c$ acting on a general smooth in space function $\phi$ is given by

$$L^c(\phi) := -\varepsilon^2 \phi_{xxx} + (f'(u^h)\phi_x)_x.$$

The results of this section are quite technical since their proof involves extensive computations related to the stationary problem (1.3) properties. The new estimated terms appear only in the stochastic setting due to the presence of noise, and where therefore not treated in the work of Bates & Xun [7, 8] or Carr & Pego [21, 22]. A main result of this paper is that the stochastic treatment of the very important deterministic result of Bates & Xun gives new insights on the analysis of the deterministic stationary problem by means of a higher order regularity point of view.

3. FRONT MOTION

In this section, we derive the equations of motions of the fronts and show that the approximate manifold is locally attracting.
3.1. Preliminaries and definitions. Let us first recall some notation. If \( u \) is the solution of (SC-H), then \( \tilde{u}(x, t) := \int_{y}^{x} u(y, t) dy \) is the solution of the integrated one i.e. of (ISC-H). Let \( a, \varepsilon, \rho, N \) be given; for some \( \ell \) such that \( \varepsilon/\ell < \rho \), we consider the unique solution \( \phi \) of (1.3) which satisfies the properties (a) and (b). Let also \( (h_{1}, \ldots, h_{N+1}) \in \Omega_{\rho} \) be the admissible interfaces positions and take \( h_{0} := -h_{1}, h_{N+2} := 2 - h_{N+1} \).

Let \( \ell_{j} = h_{j} - h_{j-1} \) be the distance between interfaces and \( \ell := \min\{\ell_{1}, \ldots, \ell_{N}\} \) the lower bound on them. Note that by the construction of \( \Omega_{\rho} \) the functions \( \phi \) are always well defined. Let

\[
\rho \ := \ v \ \partial h \ |
\]

In view of (1.3), we also define

\[
\phi^{j}(x) := \phi(x - m_{j}, \ell_{j}, (-1)^{j}),
\]

and \( u_{j} := \partial u/\partial x \) for \( j = 1, \ldots, N+1 \). Considering \( r_{j} := \varepsilon/\ell_{j} \), let

\[
\beta(\rho) := \begin{cases} 
\beta_{+}(r_{j}) & \text{for } j \text{ even}, \\
\beta_{-}(r_{j}) & \text{for } j \text{ odd}, 
\end{cases}
\]

We denote that in [8], as an application of the implicit function Theorem,

\[
(3.1) \quad \frac{\partial h_{N+1}}{\partial h_{j}} = (-1)^{N-j} + O(\varepsilon^{-1} \beta(\rho)).
\]

In addition, let

\[
\alpha(\rho) := \begin{cases} 
\alpha_{+}(r_{j}) & \text{for } j \text{ even}, \\
\alpha_{-}(r_{j}) & \text{for } j \text{ odd}, 
\end{cases}
\]

We see later, that both \( \alpha \) and \( \beta \) are exponentially small in \( \varepsilon \), if we consider \( r_{j} \leq \rho \leq \varepsilon^{\kappa} \) for some small positive \( \kappa \).

3.2. The general SDE for the front motion. Let \( \tilde{u} \) be a solution of (ISC-H). We assume that the \( N \) front positions, i.e. the coordinates of \( (\xi(t)) = (\xi_{1}(t), \ldots, \xi_{N}(t)) \), define a multi-dimensional diffusion process which is given by

\[
(3.2) \quad d\xi_{k} = b_{k}(\xi) dt + \langle \sigma_{k}(\xi), dW \rangle, \quad k = 1, \ldots, N,
\]

for some vector field \( b: \mathbb{R}^{N} \to \mathbb{R}^{N} \) and some variance \( \sigma: \mathbb{R}^{N} \to H^{N} \). The main aim of this paragraph is to identify \( b \) and \( \sigma \), which might also depend on \( \tilde{v} \), i.e. on the distance from the manifold.

We use Itô-formula, in order to differentiate \( \tilde{u}^{\xi} \) with respect to \( t \), and get

\[
(3.3) \quad d\tilde{u}^{\xi} = \sum_{j=1}^{N} \tilde{u}^{\xi}_{j} d\xi_{j} + \frac{1}{2} \sum_{1 \leq k, l \leq N} \tilde{u}^{\xi}_{kl} d\xi_{k} d\xi_{l} + d\tilde{v}, \quad \text{with } \tilde{u}^{\xi}_{kl} = \frac{\partial^{2} \tilde{u}^{\xi}}{\partial \xi_{k} \partial \xi_{l}}.
\]

We take as in [8], p. 175, the inner product in space of equation (ISC-H) with \( E_{i}^{\xi} \), to get for any \( i = 1, \ldots, N \)

\[
(3.4) \quad \langle E_{i}^{\xi}, d\tilde{u} \rangle = \langle \mathcal{L}^{\xi}(\tilde{u}), E_{i}^{\xi} \rangle dt + \langle E_{i}^{\xi}, dW \rangle,
\]

where we defined the nonlinear ICH-operator as

\[
\mathcal{L}^{\xi}(u) := -\varepsilon^{2} u_{xxxx} + (f(u_{x}))_{x}
\]

for short-hand notation.

On the other hand, if we take the inner product of (3.3) with \( E_{i}^{\xi} \), we derive

\[
(3.5) \quad \langle E_{i}^{\xi}, d\tilde{u} \rangle = \sum_{j=1}^{N} \langle \tilde{u}^{\xi}_{j}, E_{i}^{\xi} \rangle d\xi_{j} + \frac{1}{2} \sum_{1 \leq k, l \leq N} \langle \tilde{u}^{\xi}_{kl}, E_{i}^{\xi} \rangle d\xi_{k} d\xi_{l} + \langle E_{i}^{\xi}, d\tilde{v} \rangle.
\]

Throughout the rest of this paper, any summation is on \( 1, 2, \ldots, N \) for any index.
In order to eliminate $d\tilde{v}$, we apply Itô-formula to the orthogonality condition $\langle \tilde{v}, E_i^\xi \rangle = 0$, and arrive at

$$\langle E_i^\xi, d\tilde{v} \rangle = -\langle \tilde{v}, dE_i^\xi \rangle - \langle d\tilde{v}, dE_i^\xi \rangle = -\sum_j \langle \tilde{v}, E_{ij}^\xi \rangle d\xi_j - \frac{1}{2} \sum_{j,k} \langle \tilde{v}, E_{ijk}^\xi \rangle d\xi_j d\xi_k - \sum_j \langle E_{ij}^\xi, d\tilde{v} \rangle d\xi_j.$$  

(3.6)

Now, we use that $d\tilde{v} = d\tilde{u} - d\tilde{u}^\xi$ and the fact that $dt \, dt = 0$ and $dW \, dt = 0$. In details,

$$\sum_j \langle E_{ij}^\xi, d\tilde{v} \rangle d\xi_j = -\sum_j \langle E_{ij}^\xi, d\tilde{u} \rangle d\xi_j - \sum_{j,k} \langle \tilde{v}, E_{ijk}^\xi \rangle d\xi_j d\xi_k - \sum_j \langle E_{ij}^\xi, \tilde{u}^\xi \rangle d\xi_j.$$  

Therefore, by (3.6) it follows that

$$\sum_j \langle E_{ij}^\xi, d\tilde{v} \rangle d\xi_j = -\sum_j \langle \tilde{v}, E_{ij}^\xi \rangle d\xi_j - \frac{1}{2} \sum_{j,k} \langle \tilde{v}, E_{ijk}^\xi \rangle d\xi_j d\xi_k - \sum_j \langle dW, E_{ij}^\xi \rangle d\xi_j + \sum_{j,k} \langle \tilde{u}^\xi, \tilde{u}^\xi \rangle d\xi_j d\xi_k.$$  

(3.7)

Combining (3.4) with (3.5) and (3.7) we arrive at

$$\sum_j \left[ \langle \tilde{u}^\xi, E_{ij}^\xi \rangle - \langle \tilde{v}, E_{ij}^\xi \rangle \right] d\xi_j = \langle \sigma(\xi), dW \rangle dt + \sum_{j,k} \left[ \frac{1}{2} \langle \tilde{v}, E_{ijk}^\xi \rangle - \frac{1}{2} \langle \tilde{u}^\xi, E_{ijk}^\xi \rangle - \langle \tilde{u}^\xi, E_{ij}^\xi \rangle \right] d\xi_j d\xi_k$$  

$$+ \sum_j \langle dW, E_{ij}^\xi \rangle d\xi_j + \langle E_{ij}^\xi, dW \rangle.$$  

(3.8)

**Lemma 3.1.** For all $1 \leq k, l \leq N$ it holds that

$$\langle \sigma_k(\xi), dW \rangle \langle \sigma_l(\xi), dW \rangle = \langle Q \sigma_k(\xi), \sigma_l(\xi) \rangle dt.$$  

Proof. Since $d\beta_j d\beta_l = \delta_{jl} dt$ and $W(t) = \sum_{k=1}^{\infty} \alpha_k \beta_k e_k$ we obtain, using Parceval identity,

$$\langle \sigma_k(\xi), dW \rangle \langle \sigma_l(\xi), dW \rangle = \sum_{i,j} \alpha_i \alpha_j \langle \sigma_k(\xi), e_i \rangle \langle \sigma_l(\xi), e_j \rangle d\beta_j d\beta_i = \sum_j \alpha_j^2 \langle \sigma_k(\xi), e_j \rangle \langle \sigma_l(\xi), e_j \rangle dt$$  

$$= \sum_j \langle Q \sigma_k(\xi), e_j \rangle \langle \sigma_l(\xi), e_j \rangle dt = \langle Q \sigma_k(\xi), \sigma_l(\xi) \rangle dt.$$  

□

Analogously to this Lemma we easily obtain (using $dt \, dW = 0$)

$$\langle E_{ij}^\xi, dW \rangle d\xi_j = \langle E_{ij}^\xi, dW \rangle \langle \sigma_j(\xi), dW \rangle = \langle Q E_{ij}^\xi, \sigma_j(\xi) \rangle dt.$$  

Moreover, for short-hand notation, as in (7), we define the matrix $A(\xi) = (A_{ij}(\xi)) \in \mathbb{R}^{N \times N}$ by

$$A_{ij}(\xi) = \langle \tilde{u}^\xi, E_{ij}^\xi \rangle - \langle \tilde{v}, E_{ij}^\xi \rangle,$$  

which is invertible, provided that we are near the slow manifold (cf. Lemma 3.4 later). Let us denote the inverse matrix of $A$ by $A^{-1}(\xi) = (A_{ij}^{-1}(\xi)) \in \mathbb{R}^{N \times N}$.  

Therefore, for all \( i \in \{1, \ldots, N\} \) we arrive at
\[
\sum_j A_{ij}(\xi) d\xi_j = \langle \mathcal{L}^c(\dot{u}^\xi + \dot{v}), E^\xi_i \rangle dt \\
+ \sum_{l, k} \left[ \frac{1}{2} \langle \dot{v}, E^\xi_{lkl} \rangle - \frac{1}{2} \langle \dot{u}^\xi_k, E^\xi_{li} \rangle - \langle \dot{u}^\xi_k, E^\xi_{lil} \rangle \right] \langle Q\sigma_k(\xi), \sigma_l(\xi) \rangle dt \\
+ \sum_j \langle QE^\xi_{ij}, \sigma_j(\xi) \rangle dt + \langle E^\xi_i, dW \rangle.
\]
To obtain the equation for \( d\xi \) we use that \( d\xi = A(\xi)^{-1} A(\xi) d\xi \).

Thus, the final equation for \( \xi \) (as long as \( \dot{u} \) is near the manifold) is given for any \( r = 1, \ldots, N \) by
\[
d\xi_r = \sum_i A^{-1}_{ri}(\xi) \langle \mathcal{L}^c(\dot{u}^\xi + \dot{v}), E^\xi_i \rangle dt \\
+ \sum_{i, l, k} A^{-1}_{ri}(\xi) \left[ \frac{1}{2} \langle \dot{v}, E^\xi_{lkl} \rangle - \frac{1}{2} \langle \dot{u}^\xi_k, E^\xi_{li} \rangle - \langle \dot{u}^\xi_k, E^\xi_{lil} \rangle \right] \langle Q\sigma_k(\xi), \sigma_l(\xi) \rangle dt \\
+ \sum_j A^{-1}_{ri}(\xi) \langle QE^\xi_{ij}, \sigma_j(\xi) \rangle dt + \sum_i A^{-1}_{ri}(\xi) \langle E^\xi_i, dW \rangle.
\]
We can now recover \( \sigma \) and \( b \) from (3.11). The only term that does involve noise is the last one. Thus, in view of (3.2) we get
\[
\sigma_r(\xi) = \sum_i A^{-1}_{ri}(\xi) E^\xi_i.
\]
After we obtained \( \sigma \), we can proceed, in order to determine \( b(\xi) \) from the remaining terms (cf. (3.2)). So, we get for \( r = 1, \ldots, N \) that
\[
b_r(\xi) = \sum_i A^{-1}_{ri}(\xi) \langle \mathcal{L}^c(\dot{u}^\xi + \dot{v}), E^\xi_i \rangle \\
+ \sum_{i, l, k} A^{-1}_{ri}(\xi) \left[ \frac{1}{2} \langle \dot{v}, E^\xi_{lkl} \rangle - \frac{1}{2} \langle \dot{u}^\xi_k, E^\xi_{li} \rangle - \langle \dot{u}^\xi_k, E^\xi_{lil} \rangle \right] \langle Q\sigma_k(\xi), \sigma_l(\xi) \rangle \\
+ \sum_i A^{-1}_{ri}(\xi) \sum_j \langle QE^\xi_{ij}, \sigma_j(\xi) \rangle.
\]

Remark 3.2. (Well defined coordinates) It is easy to check from the construction, that given our \( \xi(t) \) from Equation (3.2) with \( b \) and \( \sigma \) defined as above, then there exists a corresponding solution \( \dot{u} \) of (ISC-H). If the distance \( \dot{u} \) from the manifold \( M \) is sufficiently small, then \( \xi \) describes the motion of the interfaces of \( \dot{u} \).

3.3. Stability and Attractivity of the manifold. In this paragraph, we prove the stability and discuss the attractivity of \( \mathcal{M} \). Considering the stability, we show that with high probability (over a long time-scale) the solution stays close to \( \mathcal{M} \), unless an interface breaks down.

In [7, Theorem B], Bates and Xun show that in the deterministic setting the slow manifold is exponentially attracting in a \( O(\varepsilon^{7/2}) \)-neighborhood in \( H^2 \), until the solution reaches an exponentially small neighborhood, where the motion of the solution along the manifold is exponentially slow. Using large deviation estimates, it is straightforward to verify for small noise, that the stochastic solution follows the deterministic one up to error terms of the order of the noise strength. Hence, the exponential attraction of \( \mathcal{M} \) still holds for (ISC-H), until the solution reaches a neighborhood of the manifold that is determined by the strength of the noise.

Here, for simplicity of presentation we will focus only on the stability of \( \mathcal{M} \). The proof can be easily modified to show attraction, too. Once, we are in the slow channel around \( \mathcal{M} \), with high probability we cannot exit for a long time-scale \( T_r \), unless one of the interfaces breaks down.
We define the metrics $A_\varepsilon$ and $B_\varepsilon$ as

\[(3.14) \quad A_\varepsilon(\tilde{v}) = \int_0^1 [\varepsilon^2 \tilde{v}_{xx}^2 + f'(u)\tilde{v}_{x}^2]dx \quad \text{and} \quad B_\varepsilon(\tilde{v}) = \int_0^1 [\varepsilon^2 \tilde{v}_{xx}^2 + \tilde{v}_{x}^2]dx.\]

Note that it is easy to check that

\[\|\partial_x \tilde{v}\|_2 \leq B_\varepsilon(\tilde{v}) \leq C\varepsilon^{-2}A_\varepsilon(\tilde{v}) \leq C\varepsilon^{-2}B_\varepsilon(\tilde{v}) \leq C\varepsilon^{-2}\|\tilde{v}\|^2_{L^2},\]

and

\[(3.15) \quad \|\tilde{v}\|_\infty^2 \leq B_\varepsilon(\tilde{v}) , \quad \|\tilde{v}_x\|_\infty^2 \leq \frac{1+\varepsilon}{\varepsilon^2}B_\varepsilon(\tilde{v}).\]

**Definition 3.3.** (cf. [7], p. 452) Define a neighborhood $\Gamma'$ of $\tilde{M}$ by

\[\Gamma' = \{\tilde{u}^\xi + \tilde{v} : \xi \in \Omega_\rho, \, B_\varepsilon(\tilde{v}) < \varepsilon^3\},\]

and we define the slow tube $\Gamma$ by

\[\Gamma := \{\tilde{u}^\xi + \tilde{v} : \xi \in \Omega_\rho, \, A_\varepsilon(\tilde{v}) < \delta_\varepsilon^{2-\kappa}\},\]

where $0 < \kappa \ll 1$ presented in the definition of the noise (cf. Assumption 2.3) and $\delta_\varepsilon$ estimates the noise strength.

The small tube $\Gamma'$ is a neighborhood of the slow manifold, where the coordinate system (cf. (1.7)) is well defined, while the slow tube $\Gamma$ is a neighborhood in which solutions with high probability do not exit for long times unless one of the interfaces breaks down. Recall that $\Gamma \subset \Gamma'$ by definition of $\delta_\varepsilon$. We even have $B_\varepsilon(\tilde{v}) < C\delta_\varepsilon^{2-\kappa}\varepsilon^{-2} \leq C\varepsilon^7$, which we need in the proof of stability.

As indicated in the introduction, the first term at the right-hand side of the flow given by (3.11), is identical to the right-hand side of the deterministic flow and has been estimated in [7]. In our stochastic case, in order to approximate the flow, we need to bound also the additional higher order terms and estimate the contribution of the noise. Later, in the next Section 4, we will identify the dominant terms in (3.11).

Using (4.27) of [8] and the fact that $\|E_{ij}^\xi\| = \mathcal{O}(\varepsilon^{-1/2})$ ([8] p. 187), we obtain in $\Gamma'$ considering the matrix $A$ the following invertibility result:

**Lemma 3.4.** Suppose that $h \in \Omega_\rho$ and $\|\tilde{v}\| = \mathcal{O}(\varepsilon^{3/2})$, then

\[A_{ij}(\xi) = \mathcal{O}(\varepsilon) + \begin{cases} (-1)^{i+j}4\ell_{j+1} & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}\]

and the matrix is invertible, with

\[A_{ij}^{-1}(\xi) = \mathcal{O}(\varepsilon) + \begin{cases} \frac{1}{\ell_j} & \text{if } i = j, j + 1 \\ 0 & \text{otherwise} \end{cases}\]

where $1 > \ell_i > \varepsilon/\rho$ denotes the length of the $i$-th interface.

As the equation is deterministically stable, we should be able to show that $\tilde{v}$ stays small for a long time (depending on the noise strength). To be more precise, we show a bound on $A_\varepsilon(\tilde{v})$ for solutions near $\tilde{M}$. Following [7] p. 449, we consider equation (3.3)

\[d\tilde{v} = d\tilde{u} - \sum_{j=1}^N \tilde{u}_j^\xi d\xi_j - \frac{1}{2} \sum_{kl} \tilde{u}_k^\xi d\xi_k d\xi_l,\]

and thus the key equation for the distance from the manifold $\tilde{M}$ is described by

\[(3.16) \quad d\tilde{v} = \mathcal{L}^\xi(\tilde{u})dt - \sum_j \tilde{u}_j^\xi b_j(\xi)dt - \sum_j \tilde{u}_j^\xi \langle \sigma_j(\xi), dW \rangle - \frac{1}{2} \sum_{kl} \tilde{u}_k^\xi (\mathbb{Q}\sigma_k(\xi), \sigma_l(\xi))dt + dW.\]

We can now proceed (cf. also (86) of [7]) and show a bound on $\tilde{v}$ in terms of $A_\varepsilon$. 


Fix some large time $T_\varepsilon$ and define $\tau^* > 0$ as the first exit time (below the threshold $T_\varepsilon$) of $\tilde{u}$ from $\Gamma'$. This is the stopping time

$$\tau^* = T_\varepsilon \wedge \inf\{ t > 0 : \xi(t) \notin \Omega_\rho \text{ or } A_\varepsilon(\tilde{v}(t)) \geq \delta_\varepsilon^{2-\kappa} \}.$$  

Note that for $t \leq \tau^*$ also $B_\varepsilon(\tilde{v}(t)) \leq C\varepsilon^{6+\kappa}$.

**Definition 3.5.** We say that a term is $O(\varepsilon_\alpha)$, if it is asymptotically smaller than any polynomial uniformly for times $t \leq \tau^*$.

Note that $\alpha, \beta$ are $O(\varepsilon_\varepsilon)$, if $\rho = \varepsilon^\kappa$.

**Theorem 3.6.** Suppose $\rho = \varepsilon^\kappa$ for some small $\kappa > 0$, $\delta_\varepsilon \geq C\varepsilon^{-q}$ for any $q > 0$, and suppose that for all $p > 0$ there exists a constant $c_\rho > 0$ such that $E A_\varepsilon(\tilde{v}(0))^p \leq C_p \delta_\varepsilon^{2p}$. Then for all $p > 0$ there exists a constant $C_p > 0$ such that

$$E A_\varepsilon(\tilde{v}(\tau^*))^p \leq C_p (T_\varepsilon + 1) \delta_\varepsilon^{2p}.$$

Therefore, we can show that the probability that the solution exits from the slow tube before $T_\varepsilon$ (i.e. $\tau^* = T_\varepsilon$) or an interface is breaking down (i.e. $\xi(\tau^*) \notin \Omega_\rho$) is bounded above by

$$\mathbb{P}(A_\varepsilon(\tilde{v}(\tau^*)) \geq \delta_\varepsilon^{2-\kappa} \leq s \mathbb{P}(\hat{\tau}) \delta_\varepsilon^{p(2-\kappa)} \leq C_p (T_\varepsilon + 1) \delta_\varepsilon^{2p}$$

for any $p > 0$. Thus the probability that the solution exits from the slow tube before $T_\varepsilon$ is of order $O(\varepsilon_\varepsilon)$ provided $T_\varepsilon \ll \delta_\varepsilon^{-q}$ for some large $q > 0$. The typical case for applications would be to consider a noise strength polynomial in $\varepsilon$, where we can take $T_\varepsilon = \varepsilon^{-q}$ for any $q > 0$.

**Remark 3.7.** (Exponentially small noise-strength $\delta_\varepsilon$) If we want to have exponentially long times $T_\varepsilon$, then we need to take exponentially small noise strength $\delta_\varepsilon$ and look closer at the various error terms in the proof of Theorem 3.6. This is straightforward, but for simplicity of presentation, we refrain from stating details here.

On the other hand, assuming that $\delta_\varepsilon$ is exponentially small, the probability of the solution exiting the slow tube $\Gamma$ before $T_\varepsilon$, without an interface breaking down, is exponentially small, even for exponentially large times $T_\varepsilon$.

### 3.4. Bounds on the sde.

The following Lemmas replace the bound on $\dot{\xi}$, which is used in the deterministic setting (cf. Lemma 4.3. in [7]).

**Lemma 3.8.** Let $\tilde{u}^\xi + \tilde{v} \in \Gamma'$ and $r = 1, \ldots, N$, then (with $E_{N+1}^\xi = 0$ for shorthand notation)

$$\sigma_r(\xi) = \frac{1}{4\ell_{r+1}} (E_{r}^\xi + E_{r+1}^\xi) + O(\varepsilon),$$

and

$$\|\sigma_r(\xi)\| \leq C/\ell < C\rho/\varepsilon.$$

**Proof.** Note that $\|\tilde{v}\| \leq B_\varepsilon(\tilde{v})^{1/2}$. Thus from the definition of $\sigma$ (cf. (3.12)), Lemma 3.4 and the bound on $E_1^\xi$ one has

$$\|\sigma_r(\xi)\| \leq \sum_{i} |A_{r-1}^{-1}(\xi)| \|E_i^\xi\| \leq C/\ell.$$

Moreover

$$\sigma_r(\xi) = A_{r-1}^{-1} E_r^\xi + A_{r+1}^{-1} E_{r+1}^\xi + O(\varepsilon),$$

and the claim follows from Lemma 3.4.

The next Lemma estimates the vector field $b$ of the diffusion process $\xi$. 

\[ \square \]
Lemma 3.9. Let \( \bar{u}^{\xi} + \bar{v} \in \Gamma' \) and assume that \( \rho = \varepsilon^{\kappa} \) for some small \( \kappa > 0 \), then there is a constant \( c > 0 \) such that

\[
|b_r(\xi)| \leq c\|Q\|\left\{ \varepsilon^{3\kappa-7/2} + \varepsilon^{2\kappa-5/2} \right\} + O(e_\varepsilon),
\]

for any \( r = 1, \ldots, N \).

Proof. We recall first \( b_r \)

\[
(3.18) \quad b_r(\xi) = \sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}_c(\bar{u}^{\xi} + \bar{v}), E_i^{\xi} \rangle \\
+ \sum_{i,l,k} A_{ri}^{-1}(\xi)\left[ \frac{1}{2}\langle \bar{u}^{\xi}, E_{il}^{\xi} \rangle - \frac{1}{2}\langle \bar{u}^{\xi}_{kl}, E_i^{\xi} \rangle - \langle \bar{u}^{\xi}_{kli}, E_i^{\xi} \rangle \right] \langle Q\sigma_k(\xi), \sigma_l(\xi) \rangle \\
+ \sum_i A_{ri}^{-1}(\xi) \sum_j \langle QE_{ij}^{\xi}, \sigma_j(\xi) \rangle.
\]

Then we use Lemma [3.4] and the bound on \( \sigma \). Moreover, in Section 5, after tedious computations the next estimates are derived (cf. [5.44], [5.45], [5.41], and [5.42], respectively):

\[
\|\langle \bar{u}^{\xi}_{kl}, E_i^{\xi} \rangle \| \leq O(\varepsilon^{-1/2})\left[ 4\ell_3 + O(\varepsilon^{-3}\beta) \right],
\]

\[
\|\langle \bar{u}^{\xi}_{k}, E_{il}^{\xi} \rangle \| \leq O(\varepsilon^{-1/2} + \varepsilon^{-4}\beta),
\]

\[
\|\langle \bar{v}, E_{il}^{\xi} \rangle \| \leq O(\varepsilon^{-3}\beta + \varepsilon^{-5}\beta) \leq \varepsilon + O(\varepsilon^{-7/2}\beta),
\]

since in the slow channel \( \|\bar{v}\| = \|\bar{v}\|_{\infty} \leq cB_\varepsilon(\bar{v})^{1/2} \leq \varepsilon^{3/2} \). Moreover,

\[
\|E_{ij}^{\xi}\| \leq 4\ell_3 + O(\varepsilon^{-3}\beta), \quad \|E_{ij}^{\xi}\| \leq O(\varepsilon^{-1/2} + \varepsilon^{-4}\beta) .
\]

In addition, we observe that (cf. [8])

\[
|\sum_i A_{ri}^{-1}(\xi)\langle \mathcal{L}_c(\bar{u}^{\xi} + \bar{v}), E_i^{\xi} \rangle | = O(\alpha/\ell) + O(\varepsilon\alpha) = O(\varepsilon).
\]

In this way, since \( \sigma = O(\rho^{-1}) \) and \( A_{ij}^{-1} = O(\rho^{-1}) \), we obtain

\[
|b_r(\xi)| \leq c\|Q\|\rho^3\varepsilon^{-3/2} + c\|Q\|\rho^2\varepsilon^{-5/2} + O(e_\varepsilon) \leq c\|Q\|\left\{ \varepsilon^{3k-7/2} + \varepsilon^{2k-5/2} \right\} + O(e_\varepsilon).
\]

3.5. Proof of Stability. Now let us turn to the proof of the Theorem 3.6. Considering the linearized C-H-operator and using It\-ô-formula we arrive at

\[
(3.19) \quad dA_\varepsilon(\bar{v}) = d\langle -L^\varepsilon \bar{v}, \bar{v} \rangle = 2\langle -L^\varepsilon \bar{v}, d\bar{v} \rangle + \langle -L^\varepsilon d\bar{v}, d\bar{v} \rangle + dR ,
\]

with

\[
dR = \int_0^1 \bar{v}_x^2 f''(u_\xi)(du_\xi)^2 \, dx + \int_0^1 \bar{v}_x f''(u_\xi) \cdot d\bar{v}_x \, dx + \int_0^1 \bar{v}_x^2 f'''(u_\xi)(du_\xi)^2 \, dx .
\]

All terms in \( R \) come from the fact that the metric \( A_\varepsilon \) itself depends on \( \xi \) through \( f'(u_\xi) \). Using It\-ô-formula and the equations (3.2) and (3.16) for \( \xi \) and \( \bar{v} \), we expand all terms

\[
dR = \sum_j \int_0^1 \bar{v}_x^2 f''(u_\xi)u_{\xi_j} \, dx \, b_j \, dt + \sum_{i,j} \int_0^1 \bar{v}_x f''(u_\xi)u_{\xi_i} \, dx \, Q\sigma_j, \sigma_i \, dt + \sum_j \int_0^1 \bar{v}_x f''(u_\xi)u_{\xi_j} \, dx \, \sigma_j, dW \\
+ \frac{1}{2} \sum_{i,j} \int_0^1 \bar{v}_x f''(u_\xi)u_{\xi_i}u_{\xi_j} \, dx \, Q\sigma_j, \sigma_i \, dt + \sum_{i,j} \int_0^1 \bar{v}_x f''(u_\xi)u_{\xi_i} \, dx \, Q\sigma_j, \sigma_i \, dt + \sum_j \int_0^1 \bar{v}_x f''(u_\xi)u_{\xi_j} \, dx \, Q\sigma_j, d\bar{v}_x (Q\sigma_j) \, dx .
\]

Now we use Theorem 5.8 in \( \xi \) variables, to obtain that \( \|u_{\xi_j}\|_{\infty} = O(\varepsilon^{-1}) \), \( \|u_{\xi_i}\|_{\infty} = O(\varepsilon^{-2}) \) and \( \|u_{\xi_j}\| = O(\varepsilon^{-1/2}) \). Moreover, by definition it holds that \( \|\bar{v}\| \leq B_\varepsilon(\bar{v}) \), so using Lemmas 3.9 and 3.8 we have
\( b_j = \mathcal{O}(\delta^2 \varepsilon^{-7/2}) \) and \( \sigma_j = \mathcal{O}(\varepsilon^{-1}) \). Finally, as \( u^\xi \) is uniformly bounded, we can bound the nonlinearity \( f \) by a constant and get
\[
dR = \mathcal{O}(B_\varepsilon(\tilde{v})\varepsilon^{-9/2}\delta^2)dt + \mathcal{O}(B_\varepsilon(\tilde{v})^{1/2}\varepsilon^{-7/2}\delta^2)dt + \langle I_R, dW \rangle.
\]

with
\[
I_R = \sum_j \int_0^1 \tilde{v}_j^2 f''(u^\xi)u_j^\xi \, dx \quad \sigma_j = \mathcal{O}(B_\varepsilon(\tilde{v})\varepsilon^{-2})
\]

As we are in the slow channel, we obtain
\[
dR = \mathcal{O}(\delta^2)dt + \langle I_R, dW \rangle.
\]
This is the crucial and only point where we need \( B_\varepsilon(\tilde{v}) = \mathcal{O}(\varepsilon^7) \), in order to estimate the 5th term of \( R \).

Now we turn to the other terms in (3.19). Lemma 3.1 gives
\[
\frac{dA_\varepsilon(\tilde{v})}{dt} = 2(-L^c\tilde{v}, \mathcal{L}^c(\tilde{u}))dt
\]
with
\[
\mathcal{L}^c(\tilde{u}) = \mathcal{L}^c(\tilde{u}^\xi + \tilde{v}) = L^c\tilde{v} + \mathcal{L}^c(\tilde{u}^\xi) + \partial_x(f_2\partial_x \tilde{v})
\]

Moreover, note that by Lemma 5.1 in [7] we have
\[
\|\partial_x(f_2\partial_x \tilde{v})\| \leq C\varepsilon^{-2}B_\varepsilon(\tilde{v}).
\]

and thus
\[
\langle -L^c\tilde{v}, \mathcal{L}^c(\tilde{u}) \rangle \leq -\|L^c\tilde{v}\|^2 + C(\varepsilon^{-2}B_\varepsilon(\tilde{v}) + \varepsilon^{-1}\alpha(r))\|L^c\tilde{v}\|
\]
\[
\leq -\frac{2}{3}\|L^c\tilde{v}\|^2 + C\varepsilon^{-2}B_\varepsilon(\tilde{v})\|L^c\tilde{v}\| + C\varepsilon^{-2}\alpha(r)^2
\]
\[
\leq -\frac{1}{2}\|L^c\tilde{v}\|^2 + C\varepsilon^{-2}\alpha(r)^2,
\]

where we used that for some constant \( a > 0 \) independent of \( \varepsilon \) and \( r \) (cf. [7], Lemma 3.2 at p. 434, and Lemma 4.2 at p. 446)
\[
(3.27)
\]

(3.28)
\[
B_\varepsilon(\tilde{v}) < C\varepsilon^{-2}A_\varepsilon(\tilde{v}) < \frac{C}{2a}\varepsilon^{-2}\|L^c\tilde{v}\|^2.
\]

Using \( B_\varepsilon(\tilde{v}) = \mathcal{O}(\varepsilon^{6+k}) \) in the slow channel, we obtain
\[
2\langle -L^c\tilde{v}, \mathcal{L}^c(\tilde{u}) \rangle \leq -\frac{1}{2}\|L^c\tilde{v}\|^2 - aA_\varepsilon(\tilde{v}) + C\varepsilon^{-2}\alpha(r)^2.
\]
Now consider the remaining four deterministic integrals. For the term in (3.22), notice that
\[ \langle L^c \tilde{v}, \tilde{u}_j^c \rangle = \langle \tilde{v}, L^c \tilde{u}_j^c \rangle = \langle \tilde{v}, \partial_x \partial_j L^b(u^c) \rangle . \]
Thus using integration by parts and Lemma 5.2 of [7] yields
\[ |\langle L^c \tilde{v}, \tilde{u}_j^c \rangle| \leq C \| \partial_x \tilde{v} \| \varepsilon^{-2} \beta(r) = \mathcal{O}(\varepsilon). \]
We use now (3.29) to arrive at
\[ \left| \sum_j \langle -L^c \tilde{v}, \tilde{u}_j^c \rangle b_j(\xi) \right| \leq C \varepsilon^{-5/2} \beta(r) B_\varepsilon(\tilde{v})^{1/2} \sup_j |b_j(\xi)| = \mathcal{O}(\varepsilon), \]
which is exponentially small in \( \varepsilon \) by Lemma 3.9. By Definition 3.5 a term is \( \mathcal{O}(\varepsilon) \), if it is asymptotically smaller than any polynomial in \( \varepsilon \) uniformly for times \( t \leq \tau^* \).

Now let us turn to (3.24). Similarly, we get
\[ |\langle -L^c \tilde{u}_i^c, \tilde{u}_j^c \rangle| = |\langle \tilde{u}_i^c, \partial_x \partial_j L^b(u^c) \rangle| \leq \| \tilde{u}_i^c \|_{L^4} \| \partial_x \partial_j L^b(u^c) \|_{\infty} \leq C \varepsilon^{-4} \beta(r) , \]
where we used Lemma 5.1 of [7] and the bound \( \| \tilde{u}_i^c \|_{L^4} = O(1) \) (cf. (5.38), for \( \beta \) bounded). Thus we obtain for the term in (3.24)
\[ \left| \sum_{ij} \langle -L^c \tilde{u}_i^c, \tilde{u}_j^c \rangle \langle Q \sigma_i(\xi), \sigma_j(\xi) \rangle \right| \leq C \varepsilon^{-4} \beta(r) \| Q \|_{L^2} = \mathcal{O}(\varepsilon). \]

For the term in (3.23) we use the bounds on \( \langle -L^c \tilde{v}, \tilde{u}_k^c \rangle \) provided by Theorem 5.47. Thus, we get
\[ |\langle L^c \tilde{v}, \tilde{u}_k^c \rangle \langle Q \sigma_k(\xi), \sigma_i(\xi) \rangle| \leq C \| Q \| \varepsilon^{-2} C \varepsilon^{-2} \beta(r) \| \tilde{v} \| = \mathcal{O}(\varepsilon) . \]

Using similar estimates and Lemma 3.8 the term in (3.25) is also \( \mathcal{O}(\varepsilon) \).

For the term in (3.26), we use the eigenfunctions \( e_k \) of \( Q \) and the uniform bound on \( f'(u^c) \), in order to obtain
\[ \text{trace}(Q^{1/2} L^c Q^{1/2}) = \sum_{k=1}^{\infty} \alpha_k^2 \langle L^c e_k, e_k \rangle \leq C \sum_{k=1}^{\infty} \alpha_k^2 B_\varepsilon(e_k) \leq C \delta^2 \varepsilon^2 . \]
This is the largest deterministic term, as the other ones are all \( \mathcal{O}(\varepsilon) \). This term comes directly from the Itô-correction of the additive noise.

Consider now Equations (3.21) - (3.26), with all deterministic integrals already estimated and include the bound on \( R \) from (3.20). For \( t \leq \tau^* \)
\[ dA_\varepsilon(\tilde{v}(t)) \leq C \delta^2 \varepsilon^2 dt - \left( \frac{1}{2} \| L^c \tilde{v} \|^2 + a A_\varepsilon(\tilde{v}) \right) dt + I, dW > , \]
where
\[ I = \sum_j 2 \langle -L^c \tilde{v}, \tilde{u}_j^c \rangle \sigma_j(\xi) - 2 L^c \tilde{v} + I_R , \]
with \( I_R = \mathcal{O}(B_\varepsilon(\tilde{v}) \varepsilon^{-3/2}) \).

In order to bound \( I \), we use (3.29), and the asymptotic formula for \( \sigma_j(\xi) \) of Lemma 3.8 combined with (54)-(55) of [7] to obtain that
\[ \langle I, QI \rangle \leq \mathcal{O}(\varepsilon) + C \| Q \| (\| L^c \tilde{v} \|^2 + B_\varepsilon(\tilde{v})^2 \varepsilon^{-4}) . \]
Now from (3.28) as in the slow channel at least \( B_\varepsilon(\tilde{v}) = \mathcal{O}(\varepsilon^6) \) we obtain \( B_\varepsilon(\tilde{v})^2 \varepsilon^{-3} \leq C \| L^c \tilde{v} \|^2 B_\varepsilon(\tilde{v}) \varepsilon^{-6} \leq C \| L^c \tilde{v} \|^2 \) and thus
\[ \langle I, QI \rangle \leq \mathcal{O}(\varepsilon) + C \| Q \| \| L^c \tilde{v} \|^2 . \]
Now we can bound powers of $A_\varepsilon$ for $t \leq \tau^*$

\[
\frac{1}{p} dA_\varepsilon(\tilde{v})^p = A_\varepsilon(\tilde{v})^{p-1} dA_\varepsilon(\tilde{v}) + \frac{p-1}{2} A_\varepsilon(\tilde{v})^{p-2} (dA_\varepsilon(\tilde{v}))^2
\]

\[
\leq C_\varepsilon^{2\delta_\varepsilon} A_\varepsilon(\tilde{v})^{p-1} dt - \left( \frac{1}{2} \| L^c \tilde{v} \|^2 + a A_\varepsilon(\tilde{v}) \right) A_\varepsilon(\tilde{v})^{p-1} dt + A_\varepsilon(\tilde{v})^{p-1} (I, dW) + \frac{p-1}{2} A_\varepsilon(\tilde{v})^{p-2} (I, QT) dt.
\]

Taking integrals up to $\tau^*$ and expectation, we easily obtain from (3.32) and (3.33) (using that the expectation of a stochastic integral is 0)

\[
\mathbb{E} A_\varepsilon(\tilde{v}(\tau^*)) + \frac{1}{2} \mathbb{E} \int_0^{\tau^*} \| L^c \tilde{v} \|^2 dt + a \mathbb{E} \int_0^{\tau^*} A_\varepsilon(\tilde{v}) dt \leq A_\varepsilon(\tilde{v}(0)) + C T_\varepsilon \delta_\varepsilon^2,
\]

and for $p \geq 2$

\[
\frac{1}{p} \mathbb{E} A_\varepsilon(\tilde{v}(\tau^*))^p + \frac{1}{2} \mathbb{E} \int_0^{\tau^*} \| L^c \tilde{v} \|^2 A_\varepsilon(\tilde{v})^{p-1} dt + a \mathbb{E} \int_0^{\tau^*} A_\varepsilon(\tilde{v})^p dt
\]

\[
\leq \frac{1}{p} \mathbb{E} A_\varepsilon(\tilde{v}(0))^p + C \delta_\varepsilon^2 \mathbb{E} \int_0^{\tau^*} A_\varepsilon(\tilde{v})^{p-1} dt + C \mathbb{E} \int_0^{\tau^*} A_\varepsilon(\tilde{v})^{p-2} \| L^c \tilde{v} \|^2 dt.
\]

Now (using $\delta_\varepsilon \geq C \varepsilon^q$) it is easy to verify by induction on $p$ that

\[
\frac{1}{p} \mathbb{E} A_\varepsilon(\tilde{v}(\tau^*))^p + \frac{1}{2} \mathbb{E} \int_0^{\tau^*} \| L^c \tilde{v} \|^2 A_\varepsilon(\tilde{v})^{p-1} dt + a \mathbb{E} \int_0^{\tau^*} A_\varepsilon(\tilde{v})^p dt \leq C(T_\varepsilon + 1) \delta_\varepsilon^2.
\]

This implies the claim.

4. Motion of the Interfaces

In this section, we investigate in detail what the SDE (2.5) for $\xi$ actually implies for the motion of the interfaces considering some important special cases, where the equation is simplified a lot. We assume first that the noise is exponentially small. Then considering the two interfaces problem (i.e. when $N = 1$) we discuss the case of noise strength being polynomial in $\varepsilon$. Finally, although not covered by our theorems, we present some comments on how the equation would look like for non-smooth in space space-time white noise, which means that $Q$ is the identity.

Let us first state the result we achieved so far. The motion of the interfaces for the stochastic model is given by the following theorem.

**Theorem 4.1.** Let $\tilde{v}^c + \tilde{v} \in \Gamma'$ and assume that $\rho$ is small, then the equations dominating the flow of the Stochastic Cahn-Hilliard equation within the slow channel are given by

\[
d\xi_1 = \frac{1}{4 \ell_2} (\alpha^3 - \alpha^1) dt + O(\varepsilon \alpha) dt + dA^{(1)}
\]

\[
d\xi_2 = \frac{1}{4 \ell_2} (\alpha^3 - \alpha^1) dt + \frac{1}{4 \ell_3} (\alpha^4 - \alpha^2) dt + O(\varepsilon \alpha) dt + dA^{(2)}
\]

\[
d\xi_3 = \frac{1}{4 \ell_3} (\alpha^4 - \alpha^2) dt + \frac{1}{4 \ell_4} (\alpha^5 - \alpha^3) dt + O(\varepsilon \alpha) dt + dA^{(3)}
\]

\[
\vdots
\]

\[
d\xi_N = \frac{1}{4 \ell_N} (\alpha^{N+1} - \alpha^N) dt + \frac{1}{4 \ell_{N+1}} (\alpha^{N+2} - \alpha^N) dt + O(\varepsilon \alpha) dt + dA^{(N)},
\]

where

\[
\alpha^j = \frac{1}{2} K_\pm^2 A_\pm^2 \exp(-A_\pm \ell_j / \varepsilon) \left[ 1 + O\left( \frac{\ell_j}{\varepsilon} \exp\left( \frac{-A_\pm \ell_j}{2 \varepsilon} \right) \right) \right], \quad j = 1, 2, \ldots, N + 2,
\]
for
\begin{align}
A_\pm := f'(\pm 1) \quad \text{and} \quad K_\pm := 2\exp\left[ \int_0^1 \frac{A_\pm}{2F(\pm t)^{1/2}} - \frac{1}{1-t} \right] dt.
\end{align}

Here, the stochastic processes \(A^{(r)}_s\), \(r = 1, \ldots, N\) are related to the noise; they depend on the symmetric operator \(Q\) and the variance \(\sigma\), and are given by the formula
\begin{align}
dA^{(r)}_s := \sum_{i, l, k} A^{-1}_{ri} (\xi) \left[ \frac{1}{2} (\tilde{v}, E^\xi_{ilk}) - \frac{1}{2} (\tilde{u}^\xi_k, E^\xi_{il}) - \langle \tilde{u}^\xi_k, E^\xi_{il} \rangle \right] (Q\sigma_k (\xi), \sigma_l (\xi)) dt
+ \sum_i \sum_{l, k} A^{-1}_{ri} (\xi) \langle Q E^\xi_{ij}, \sigma_j (\xi) \rangle dt + \sum_i \sum_{l, k} A^{-1}_{ri} (\xi) (E^\xi_{il}, dW).
\end{align}

Proof. Remind that as long as \(\tilde{u}\) is near the manifold, then by (3.11) we obtained for any \(r = 1, \ldots, N\)
\begin{align}
d\xi_r = \sum_i A^{-1}_{ri} (\xi) \langle Q^c (\tilde{u}^\xi + \tilde{v}), E^\xi_j \rangle dt + dA^{(r)}_s.
\end{align}

Lemma 3.4 gives that the matrix \(A^{-1}\) and therefore the terms \(\sum_i A^{-1}_{ri} (\xi) \langle Q^c (\tilde{u}^\xi + \tilde{v}), E^\xi_j \rangle\) are identical to those presented in \([7, 8]\) for the deterministic case (i.e. when \(dA^{(r)}_s = 0\) for any \(r\)). Hence, using (4.32) of \([8]\) we obtain the result. \(\square\)

Remark 4.2. Denote that using the relation \(\ell_j = h_j - h_{j-1}\) and the asymptotic formula for \(\frac{\partial h_{N+1}}{\partial t}\) we can derive an analogous system in \(h_j\) or in \(\ell_j\) (cf. \([8]\)).

We observe that
\begin{align}
dA^{(r)}_s := A^{(r)}_Q dt + \sum_i A^{-1}_{ri} (\xi) \langle E^\xi_j, dW \rangle ,
\end{align}
for
\begin{align}
A^{(r)}_Q := \sum_i \sum_{l, k} A^{-1}_{ri} (\xi) \left[ \frac{1}{2} (\tilde{v}, E^\xi_{ilk}) - \frac{1}{2} (\tilde{u}^\xi_k, E^\xi_{il}) - \langle \tilde{u}^\xi_k, E^\xi_{il} \rangle \right] \langle Q\sigma_k (\xi), \sigma_l (\xi) \rangle
data + \sum_i \sum_{l, k} A^{-1}_{ri} (\xi) \langle Q E^\xi_{ij}, \sigma_j (\xi) \rangle.
\end{align}

Following Lemma 3.9 we obtain in the slow channel that
\begin{align}
|A^{(r)}_Q| \leq c\|Q\|^2 \rho \varepsilon^{-3/2} + \varepsilon^{-5/2}, \quad \text{for all} \quad r = 1, \ldots, N.
\end{align}

Thus, in case of \(\|Q\| = O(\varepsilon^{1/2})\), since \(\rho\) is at least bounded, we can show that \(A^{(r)}_Q = O(\varepsilon\alpha)\). It is not hard to show that we can also neglect the stochastic term from (4.1), in order to recover the result of Bates & Xun on metastable slow motion, at least with high probability.

An interesting case arises, when the additional terms in \(A^{(r)}_Q\) are of order \(O(\alpha)\). Then we obtain additional terms in (4.1). Nevertheless, for simplicity of presentation, we refrain from stating details here.

4.1. Polynomial noise strength. For the remainder of this section we fix \(N = 1\), which is the case of two interfaces, and a noise strength \(\delta_c = \varepsilon^\delta\) for some \(\delta > 9/2\). To be more precise suppose \(Q = Q_0 \varepsilon^\delta\) with \(Q_0 = O(1)\).

Using (4.1), we notice that the equation of motion for the first interface is given by
\begin{align}
d\xi_1 = O(\alpha) dt + dA^{(1)}_s,
\end{align}
and the motion of the second interface is fixed due to mass conservation.

Recall that \(\ell_2\) is the distance between the two interfaces, and fix \(\rho = \varepsilon^\kappa\), which means that the lower bound on \(\ell_2\) is \(\varepsilon^{1-\kappa}\). Let us now first look at (3.12)
\begin{align}
\sigma_1 (\xi) = A^{-1}_{11} E^\xi_1.
\end{align}
Since \( \tilde{u}_1^\xi = \tilde{u}_1 + \frac{\partial u_1}{\partial n_1} \) while \( \frac{\partial u_1}{\partial n_1} = 1 + O(e_2) \) and \( E_1^\xi = \tilde{u}_1^\xi + \tilde{u}_1^\xi + O(e_2) \), it follows that

\[
E_1^\xi = \tilde{u}_1^\xi + O(e_2),
\]

and again the error term remains of the same order under differentiation w.r.t. \( \xi \). Secondly, from (4.24) in [8] there is a constant \( c_\star \) such that \( \|\tilde{u}_1^\xi\|^2 = 4\ell_2 + c_\star e_2 + O(e_2) \), and the error term remains \( O(e_2) \) under differentiation. (In our case \( N = 1 \) we have that \( \tilde{u}_1 \) used in [8] is up to errors of order \( O(e_2) \) equal to \( \tilde{u}_1^\xi \)).

Moreover, by definition

\[
A_{11}(\xi) = \langle \tilde{u}_1^\xi, E_{11}^\xi \rangle - \langle \tilde{v}, E_{11}^\xi \rangle = \|\tilde{v}\|^2 + \|\tilde{v}\|\|O(\varepsilon^{-1/2}) + O(e_2)\|
\]

where the same estimate is used, though never presented analytically) for \( E_{11}^\xi = O(\varepsilon^{-1/2}) \). Recall that in the slow channel \( \Gamma \) we have

(4.8)

\[
\|v\|_\infty \leq (B_\varepsilon(v))^{1/2} \leq C\varepsilon^{-1}(A_\varepsilon(v))^{1/2} \leq C\varepsilon^{-1}(\varepsilon^{-2-\kappa})^{1/2} \leq C\varepsilon^{-1+\delta(1-\kappa/2)}.\]

Thus we proved

(4.9) \( A_{11} = 4\ell_2 + c_\star e_2 + O(\varepsilon^{(1-\kappa/2)-\frac{3}{2}}) \) and \( \sigma_1(\xi) = \frac{1}{4\ell_2 + c_\star e_2 + O(\varepsilon^{(1-\kappa/2)-\frac{3}{2}})}\|\tilde{A}_1^\xi\| + O(e_2) \).

Now we can consider the deterministic drift

\[
A_{Q}^{(1)} = A_{11}^{-1}(\xi) \left[ \frac{1}{2}\langle \tilde{v}, E_{11}^\xi \rangle - \langle \tilde{u}_1^\xi, E_{11}^\xi \rangle - \langle \tilde{v}, E_{11}^\xi \rangle \right] \langle Q\sigma_1(\xi), \sigma_1(\xi) \rangle + A_{11}^{-1}(\xi)\langle Q\xi_1^\xi, \sigma_1(\xi) \rangle
\]

Thus, in the slow channel \( \Gamma \) (cf. (4.8)) the equation of motion for the interface is reduced to

\[
d\xi = A_{11}^{-1}(\xi)\langle Q(\varepsilon^{(1-\kappa/2)-\frac{5}{2}})\|Q^{1/2}E_1^\xi\|dt - \frac{3}{4}A_{11}^{-3} \left( \frac{\partial}{\|E_1^\xi\|^2} \right)\|Q^{1/2}E_1^\xi\| dt + A_{11}^{-2} \frac{\partial}{\|E_1^\xi\|^2} \|Q^{1/2}E_1^\xi\| dt + A_{11}^{-1} \langle E_1^\xi, dW \rangle + O(e_2) dt.
\]

By (45) of [7] we know that

\[
\tilde{u}_1^\xi = 1 - u^\xi + O(e_2) \quad \text{and} \quad u_1^\xi = -u_1^\xi + O(e_2),
\]

(as \( [0, 1] = I_1 \cup I_2 \) and \( u^\xi(m_1) = u_1^\xi(0) = -1 + O(e_2) \)). They also proved, that the error terms remain \( O(e_2) \), under differentiation w.r.t. \( \xi \). Thus, we obtain

\[
\|\tilde{u}_1^\xi\|^2 = \|1 - u^\xi\|^2 + O(e_2) = 1 - 2M + \|u^\xi\|^2 + O(e_2).
\]

Taking again a derivative yields

\[
\frac{\partial}{\partial \xi} \|E_1^\xi\| \| = 2\langle u_1^\xi, u^\xi \rangle + O(e_2) = -2\langle u_1^\xi, u^\xi \rangle + O(e_2) = u^\xi(0)^2 - u^\xi(1)^2 + O(e_2) = O(e_2).
\]

And thus we verified that

\[
\frac{\partial}{\partial \xi} \|E_1^\xi\| = O(e_2).
\]

Therefore, the equation of motion for \( \xi \) is simplified as follows:

(4.10) \( d\xi = O(\varepsilon^{(3-\kappa/2)-\frac{11}{2}}) dt + A_{11}^{-1} \frac{\partial}{\|E_1^\xi\|^2} \|Q^{1/2}E_1^\xi\| dt + A_{11}^{-1} \langle E_1^\xi, dW \rangle \).

Although this is not covered by our assumptions, as a final example we consider a space-time white noise with \( Q = \varepsilon^\delta Id \). In this case

\[
d\xi = O(\varepsilon^{3\delta-7/2}) dt + \varepsilon^\delta A_{11}^{-1} \langle E_1^\xi, dW \rangle.
\]
which is a rescaled equation valid on the timescale $O(\varepsilon^{-3})$. Up to the small deterministic error terms, $\xi$ is a stochastic process with mean zero and quadratic variation. More specifically,

$$
\int_0^t \varepsilon^{2\delta} A_{11}^{-2} \langle E_1^\xi, E_1^\xi \rangle dt = \varepsilon^{2\delta} \int_0^t A_{11}^{-2} \|u_1\|_2^2 dt + O(\varepsilon) t
$$

$$
= \varepsilon^{2\delta} \int_0^t A_{11}^{-1} dt + O(\varepsilon^{5-3/2+\kappa}) \frac{t}{\ell_2^3} = \frac{\varepsilon^{2\delta}}{4 \ell_2^3} t + O(\varepsilon^{2\delta+1}) t + O(\varepsilon^{3\delta-7/2+\kappa}) t,
$$

which means (comparing to Levy’s characterization of Brownian motion) that in first approximation for times not too large the interface behaves similar to a Brownian motion with variance $\varepsilon^{2\delta}/(4 \ell_2^3)$.

5. Higher order estimates

5.1. Preliminaries. This section deals with the estimation of all the following higher order terms that appear due to stochastic integration when deriving the equations of motion in the slow channel:

$$
\langle \dot{v}, E_1^\xi \rangle, \quad \langle \ddot{u}_k^\xi, E_1^\xi \rangle, \quad \langle \ddot{u}_k^\xi, E_1^\xi \rangle.
$$

In addition, we bound the quantity $\langle L^\varepsilon \dot{v}, u_k^\xi \rangle$. Considering a general smooth in space function $\phi$, the operator $L^\varepsilon$ is given by

$$
L^\varepsilon(\phi) := -\varepsilon^2 \phi_{xxx} + (f'(u^h)\phi_x)_x.
$$

In order to achieve a rigorous estimation for all these terms, we investigate the properties of the stationary problem \(1.3\). Our analysis admits extensive computations and is based on the ideas and techniques presented in \[21\, \[22\, \[7\, \[8\] for the deterministic case where analogous terms of lower order have been estimated.

Denote first, that for the construction of the approximate manifold of solutions for the stochastic Cahn-Hilliard we use a local coordinate system when presenting the admissible interface positions. The $h_{N+1}$ variable depends on $h_i = \xi_i$, $i = 1, \ldots, N$, therefore, when differentiating two times in $\xi$ variables and applying the chain rule the second order term $\partial^2 h_{N+1}/\partial h_i^2$ appears. More specifically, for a general function $f$ smooth in space and any $i, j = 1, \ldots, N$, we obtain

\[
\frac{\partial f}{\partial \xi_i} = \frac{\partial f}{\partial h_i} + \frac{\partial f}{\partial h_{N+1}} \frac{\partial h_{N+1}}{\partial h_i}, \quad \text{and}
\]

\[
\frac{\partial^2 f}{\partial \xi_i \partial \xi_j} = \frac{\partial^2 f}{\partial h_i \partial h_j} + \left( \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j} + \frac{\partial^2 h_{N+1}}{\partial h_i^2} \frac{\partial h_{N+1}}{\partial h_j} \right) \frac{\partial h_{N+1}}{\partial h_i} + \frac{\partial f}{\partial h_{N+1}} \left( \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j} + \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_{N+1}} \frac{\partial h_{N+1}}{\partial h_j} \right).
\]

By the next lemma considering $\rho = \varepsilon^\kappa$ for some small $\kappa > 0$ and thus $\alpha, \beta$ are exponentially small, we estimate $|\partial^2 h_{N+1}/\partial h_i \partial h_j|$. As in \[7\], where the analogous first order estimate has been derived, we use an implicit function theorem argument combined with the mass conservation constraint. If $u^h$ is in the second approximate manifold $\mathcal{M}$ then by definition mass conservation holds i.e.

$$
M = M(h) = \int_0^1 u^h(x) dx.
$$

Differentiating two times in $h$ variables, we get

$$
\frac{d^2}{dh_i dh_j} M(h) = \int_0^1 u^h_{ij} dx,
$$

where $u^h_{ij} := \frac{\partial^2 u^h}{\partial h_i \partial h_j} = \frac{\partial u^h}{\partial h_j}$. 
Lemma 5.1. For any $i,j = 1, \ldots, N$ the next inequality follows

$$\left| \frac{\partial^2 h_{N+1}}{\partial h_i \partial h_j} \right| \leq O(\varepsilon).$$

Proof. Consider $\ell$ a generic positive variable. According to the analysis presented in [21], when comparing the $x$ and $\ell$ derivatives of the solution $\phi$ of the stationary problem (1.3), we obtain a residual function $w$ given by the following relation

(5.2) $2\phi_x(x, \ell; \pm 1) = -(\text{sgn } x) \phi_x(x, \ell; \pm 1) + 2w(x, \ell; \pm 1)$.

Let us define $I_j := [m_j, m_{j+1}]$, $\chi_j^l(x) := \chi\left(\frac{x-h_j}{\varepsilon}\right)$. If $w_j^l(x) := w(x-m_j, h_j-h_{j-1}, (-1)^j)$, then the interval $[h_{j-1} - \varepsilon, h_{j+1} + \varepsilon]$ contains the support of $u_j^h$ and

(5.3) $u_j^h(x) = \begin{cases} 
\chi_j^{l-1}w_j^l(1 - \chi_j^l)(-\phi_j^l + w_j^l) + \chi_j^l(-\phi_j^{l+1} - w_j^{l+1}) + \chi_j^{l+1}(\phi_j^{l+1} - \phi_j^l) & \text{for } x \in I_{j-1} \\
(1 - \chi_j^l)w_j^l & \text{for } x \in I_j \\
-(1 - \chi_j^{l+1})w_j^{l+1} & \text{for } x \in I_{j+1}
\end{cases}$

where $\chi_j^l = \partial_x(\chi\left(\frac{x-h_j}{\varepsilon}\right))$ and $\phi_j^l = \phi_x(x - m_j, l_j - l_{j-1}, (-1)^j)$ (cf. [21], p. 561). We denote that in $I_j$ (cf. [21] p. 430)

$u_j^h = -u_j^h + (1 - \chi_j^l)w_j^l - \chi_j^{l+1}w_j^{l+1}$

and thus

(5.4) $u_j^{hi} = -\frac{\partial u_j^h}{\partial h_i} + (\delta_j,i,\chi_j^l)w_j^l + (1 - \chi_j^l)(A_j,i,w_j^l + B_j,i,w_j^\ell)
\quad - \delta_j,i,\chi_j^{l+1}w_j^{l+1} - \chi_j^l(A_j+1,i,w_j^{l+1} + B_j+1,i,w_j^{l+1}), \text{ in } I_j$

where $w_j^l = w_x(x - m_j, l_j - l_{j-1}, (-1)^j)$, $w_j^\ell = w_x(x - m_j, l_j - l_{j-1}, (-1)^j)$, $\delta_j,i$ is the Kronecker delta, while

$A_j,i := \frac{\partial(x - m_j)}{\partial h_i} = \begin{cases} 0 & \text{for } i \neq j, j-1 \\
-1/2 & \text{for } i = j, j-1
\end{cases}$

and

$B_j,i := \frac{\partial(h_j - h_{j-1})}{\partial h_i} = \begin{cases} 0 & \text{for } i \neq j, j-1 \\
1 & \text{for } i = j \\
-1 & \text{for } i = j - 1.
\end{cases}$

In a similar way we obtain

(5.5) $u_j^{hi} = \delta_j,i,\chi_j^{l-1}w_j^l + \chi_j^{l-1}(A_j,i,w_j^l + B_j,i,w_j^\ell)$, \text{ in } I_{j-1},$

(5.6) $u_j^{hi} = \delta_j+1,i,\chi_j^{l+1}w_j^{l+1} - (1 - \chi_j^{l+1})(A_j+1,i,w_j^{l+1} + B_j+1,i,w_j^{l+1})$, \text{ in } I_{j+1}.$

Using now the estimates of $w_j^l$, $w_j^\ell$, $w_j^\ell$ (cf. [21], or [21] at p. 172), then for $r > 0$ sufficiently small, we obtain

$$\left| \int_{I_{j-1} \cup I_{j+1}} u_j^{hi}(x) \, dx \right| \leq C\varepsilon^{-2}(r^{-1} + 1)\beta(r)K_{j,i} + O(\varepsilon)\delta_{j,i},$$

with $K_{j,i} = |A_j,i| + |A_{j+1,i}| + |B_{j,i}| + |B_{j+1,i}|$ and

$$\left| \int_{I_j} \left[-\delta_j,i,\chi_j^l(x)w_j^l + (1 - \chi_j^l)(A_j,i,w_j^l + B_j,i,w_j^\ell)
\quad - \delta_j,i,\chi_j^{l+1}w_j^{l+1} - \chi_j^l(A_j+1,i,w_j^{l+1} + B_j+1,i,w_j^{l+1})\right] \, dx \right| \leq C\varepsilon^{-2}(r^{-1} + 1)\beta(r)K_{j,i} + O(\varepsilon)\delta_{j,i}.$$
Therefore, using the estimates for \( w^i \) it follows that
\[
\frac{d^2}{dh^1 dh_i} M(h) = \int_0^1 w_{ji}^h dx = \int_{I_j} -\frac{\partial^2 w^h}{\partial x \partial h_i} dx + O(\varepsilon^{-2}(r^{-1} + 1)\beta(r)) \mathcal{K}_{j,i} + O(\varepsilon) (\delta_{j-1,i} + \delta_{j,i} + \delta_{j+1,i})
\]
\[
= \int_{I_j} (\frac{\partial w^h}{\partial x}) dx + O(\varepsilon^{-2}(r^{-1} + 1)\beta(r)) \mathcal{K}_{j,i} + O(\varepsilon) (\delta_{j-1,i} + \delta_{j,i} + \delta_{j+1,i})
\]
\[
= -(w^h_{j+1} - w^h_{j}) + O(\varepsilon^{-2}(r^{-1} + 1)\beta(r)) \mathcal{K}_{j,i} + O(\varepsilon) (\delta_{j-1,i} + \delta_{j,i} + \delta_{j+1,i}).
\]
Since the support of \( w^h \) is \( I_{i-1} \cup I_i \cup I_{i+1} \ni m_{i-1}, m_i, m_{i+1}, m_{i+2} \) then we get that \( \frac{d^2}{dh^1 dh_i} M = 0 \) if \( j \neq i-1, i, i+1, i+2 \), while for example \( w^h(m_i) = \chi_i w^i |m_i = \chi_i^{-1} |m_i w(0, l_i, \pm 1) \) and \( w^h(m_{i+1}) = -(1-\chi^{i+1})w^i |m_{i+1} = -(1-\chi^{i+1})w(0, l_{i+1}, \pm 1) \). But \( w(0) = O(\varepsilon^{-1} \alpha_L') \), \([21]\) p. 558, since \( \phi_{xx}(0)^{-1} = \varepsilon^2 / W'(\phi(0)) \) and \( \varepsilon/l \) is uniformly bounded, while \( \chi \) is \( C^\infty \).

Let us now for simplicity consider \( N = 1 \) then \( M(h_1, y) = \text{constant} \), when \( y = h_2 \) where \( h_2 \) is a function of \( h_1 \), so
\[
\frac{\partial M}{\partial h_1} + \frac{\partial M}{\partial y} \frac{\partial y}{\partial h_1} = 0
\]
and thus
\[
\frac{\partial^2 M}{\partial h_1 dh_1} + (\frac{\partial M}{\partial y} \frac{\partial y}{\partial h_1} + \frac{\partial M}{\partial h_1} \frac{\partial^2 y}{\partial h_1^2}) = 0.
\]
We set \( y = h_2 \) to get using the estimate \( \frac{\partial h_{x \times y}}{\partial y} = O(1) \)
\[
O(\varepsilon) + O(\varepsilon) O(1) + O(1) \frac{\partial^2 h_2}{\partial h_1^2} = 0
\]
and thus
\[
\frac{\partial^2 h_2}{\partial h_1^2} = O(\varepsilon).
\]
The same follows when \( N > 1 \). Therefore, we obtain the result. \( \square \)

5.2. **The estimates.** We define \( I_s := [-\ell/2 - \varepsilon, \ell/2 + \varepsilon] \), then for any \( x \in I_s \) it holds that \((5) [21] [22]\)
\[
|w| \leq c \varepsilon^{-1} \beta_\pm(\ell),
\]
\[
|w_x| \leq c \varepsilon^{-2} r^{-1} \beta_\pm(\ell),
\]
\[
|w_{x\ell}| \leq c \varepsilon^{-2} \beta_\pm(\ell),
\]
\[
|w_{x\ell}| \leq c \varepsilon^{-3} r^{-1} \beta_\pm(\ell),
\]
\[
|w_{xx}| \leq c \varepsilon^{-3} \beta_\pm(\ell).
\]
(5.7)

For the purposes of our proof we will need estimates for the terms
\[
|w_{\ell\ell}|, |w_{xxx}|, |w_{x\ell\ell}|, |w_{x\ell\ell}|, |w_{x\ell\ell}|, |w_{x\ell\ell}|.
\]
It is sufficient to estimate the above terms in \( I := [0, \ell/2 + \varepsilon] \) or in \( (0, \ell/2 + \varepsilon] \). We write \( I = [0, \ell/2 - \varepsilon H] \cup [\ell/2 - \varepsilon H, \ell/2 + \varepsilon] \), for a positive \( H \) to be defined in the sequel. We set
\[
I_H := [0, \ell/2 - \varepsilon H], \quad \text{and} \quad J := [\ell/2 - \varepsilon H, \ell/2 + \varepsilon],
\]
and prove the next lemma related to the second derivative of \( w \) in \( \ell \).

**Lemma 5.2.** For any \( x \in I_s \) it holds that
\[
|w_{\ell\ell}| \leq c \varepsilon^{-3} \beta_\pm(\ell).
\]
Proof. Motivated by the proof of [22] for the estimate of \(|w_\ell|\), we use that
\[
\varepsilon^2 w_{xx} = f'(\phi(x))w \quad \text{in} \quad (0, \ell/2 + \varepsilon) \supset I_\ell^\varepsilon,
\]
and differentiate two times in \(\ell\) to obtain
\[
\varepsilon^2 (w_\ell)_xx - f'(\phi)w_\ell = \mathcal{F}
\]
for \(\mathcal{F} := f'''(\phi)\phi^2 w + f''(\phi)\phi_\ell w + 2f''(\phi)\phi_\ell^2 w\). By maximum principle it follows that
\[
|w_\ell(x)| \leq \max \left\{ |w_\ell(0)|, |w_\ell(\ell/2 - \varepsilon H)|, \sup_{x \in I_\ell^\varepsilon} \left| \mathcal{F}/f'(\phi) \right| \right\} \quad \text{for any } x \in I_\ell^\varepsilon.
\]
Following Carr and Pego (cf. [21] p. 560), we choose \(\alpha\) and \(H\) such that \(f'(\phi(x)) \geq c_0 > 0\) for \(0 < x < \ell/2 - \varepsilon H\). Since \(\varepsilon^2 \phi_\ell^2 = 2(F(\phi) - \alpha)\), then there exists \(C > 0\) such that \(\frac{1}{|\phi|} \leq \frac{1}{\varepsilon} \) for any \(x \in J = [\ell/2 - \varepsilon H, \ell/2 + \varepsilon]\) (cf. [21] p. 560, and p. 557).

We will estimate first, \(|w_\ell(x, \ell, -1)|\) in \(J\). It holds that (cf. [21] p. 558)
\[
w(x, \ell, -1) = \varepsilon^{-1} \ell^{-2} \alpha' (r) \phi_x |x|, \ell, -1 \int_{\ell/2}^{x} \frac{ds}{\phi_x (s, \ell, -1)^2}.
\]
Let us define \(A := \int_{\ell/2}^{x} \frac{ds}{\phi_x (s, \ell, -1)^2}\); for simplicity we shall refer to \(\alpha_-'\) by using the symbol \(\alpha\). We differentiate relation (5.10) and arrive at
\[
w_\ell = \varepsilon^{-1} \left\{ (\ell^{-2} \alpha'(r))_t \phi_x A + 2(\ell^{-2} \alpha'(r))_t \phi_x A + 2(\ell^{-2} \alpha'(r))_t \phi_x A_t + (\ell^{-2} \alpha'(r)) \phi_x A_{tt} \right\}.
\]
According to [21, 22] it follows that
\[
|\alpha'| \leq cr^{-2} \alpha, \quad |\alpha''| \leq cr^{-4} \alpha,
\]
and hence we obtain
\[
|\alpha'''| \leq cr^{-6} \alpha.
\]
So, observing that that \(r = \varepsilon/\ell\) is bounded, i.e. \(\ell^{-1} \leq \varepsilon^{-1}\), we get
\[
|\ell^{-2} \alpha'(r)| \leq c \varepsilon^{-2} \alpha, \quad |(\ell^{-2} \alpha'(r))_t| \leq c \varepsilon^{-3} \alpha, \quad |(\ell^{-2} \alpha'(r))_{tt}| \leq c \varepsilon^{-4} \alpha.
\]
Obviously since \(x \in J\) then \(|A| \leq \varepsilon^{2+1}\). Denote that
\[
\varepsilon^2 \phi_\ell^2 = 2(F(\phi) - \alpha)
\]
(cf. [21] p. 552), while
\[
\varepsilon^2 \phi_{xx} = f(\phi).
\]
Since \(\int_{-\ell/2}^{\ell/2} |\phi_x| \leq 2\) (cf. [21] p. 558), and \(\phi\) satisfies a Dirichlet problem then by trace inequality we get that \(\phi\) is uniformly bounded. Therefore, we obtain
\[
|\phi_x| \leq c \varepsilon^{-1}, \quad |\phi_{xx}| \leq c \varepsilon^{-2}, \quad |\phi_{xxx}| \leq c \varepsilon^{-3}.
\]
Using now the definition (5.2) of \(w\), and the fact that \(|w| + |\phi_x| \leq c \varepsilon^{-1}\), we arrive at
\[
|\phi_\ell| \leq c \varepsilon^{-1},
\]
while \(|\phi_x| \leq c |\phi_{xx}| + c |w_\ell|\). So, using that \(|w_\ell| \leq c \varepsilon^{-2}\), [8], we get
\[
|\phi_x| \leq c \varepsilon^{-2}.
\]

By (5.14) it follows that
\[
|\phi_{xx}| \leq c \varepsilon^{-3}.
\]
Finally, we will also need an estimate for the term $\phi_{x\ell\ell}$. We differentiate two times in $\ell$ the equation (5.13) and obtain

$$|\varepsilon^2 \phi_{x\ell\ell}| \leq c\varepsilon^2,$$

hence using that in $J$ it holds that $\frac{1}{|w_\ell|} \leq c\varepsilon$ we get

$$|\phi_{x\ell\ell}| \leq c\varepsilon^{-3} \text{ in } J.$$

In order to compute the derivatives of $A$ in (5.11), we apply the formulae

$$\frac{d}{d\ell} \int_{s(\ell)}^{b} g(s, \ell)ds = \int_{s(\ell)}^{b} g_\ell(s, \ell)ds - s'(\ell)g(s(\ell), \ell),$$

$$\frac{d^2}{d\ell^2} \int_{s(\ell)}^{b} g(s, \ell)ds = \int_{s(\ell)}^{b} g_\ell(s(\ell), \ell)ds - s'(\ell)g_\ell(s(\ell), \ell) - s''(\ell)g(s(\ell), \ell) - s'(\ell)^2 g_{\ell\ell}(s(\ell), \ell) - s'(\ell)g_\ell(s(\ell), \ell).$$

After tedious computations, using the above estimates and the fact that the interval’s length is of order $O(\varepsilon)$ we arrive at

$$|A_\ell| \leq c\varepsilon^2, \quad |A_{\ell\ell}| \leq c\varepsilon.$$

We denote that $\varepsilon/\ell$ is bounded i.e. $\ell^{-1} \leq c\varepsilon^{-1}$, thus by (5.11) and (5.12) we obtain

(5.15) \hspace{1cm} |w_\ell| \leq c\varepsilon^{-3} \alpha \text{ in } J.

So by (5.15), since $\ell/2 - \varepsilon H \in J$, it follows that

(5.16) \hspace{1cm} |w_\ell(\ell/2 - \varepsilon H)| \leq c\varepsilon^{-3} \alpha.

By the definition of $F$, the fact that $f' \geq c_0 > 0$ in $I_H$ and the first and third estimate of (5.7) we get for $\beta := \beta_-$ that

$$\sup_{x \in I_H} \left| \frac{F}{f}(\phi) \right| \leq c \left[ |\phi_\ell|^2 |w| + |\phi_{\ell\ell}| |w| + |\phi_\ell| |w_\ell| \right] \leq c\varepsilon^{-1} \beta \left[ |\phi_\ell|^2 + |\phi_\ell| + \varepsilon^{-1} |\phi_\ell| \right].$$

In addition, since $|w_\ell| + |\phi_{x\ell}| \leq c\varepsilon^{-2}$ [21, 8], then it follows that

$$|\phi_\ell| \leq c\varepsilon^{-2},$$

while, as we proved, $|\phi_\ell| \leq c\varepsilon^{-1}$, so

(5.17) \hspace{1cm} \sup_{x \in I_H} \left| \frac{F}{f}(\phi) \right| \leq c\varepsilon^{-3} \beta.

What is missing is the estimate of $|w_\ell(0)|$; in [22] by use of the relation $w(0) = -\frac{\partial \beta}{\partial \varepsilon}(\varepsilon/\ell)$, it was demonstrated that $|w_\ell(0)| \leq c\varepsilon^{-2} \beta$, analogously by differentiating in $\ell$ it follows that

(5.18) \hspace{1cm} |w_\ell(0)| \leq c\varepsilon^{-3} \beta.

Using now (5.9), (5.15), (5.16), (5.17) and (5.18) we obtain that $|w_\ell| \leq c\varepsilon^{-3} \beta$ for any $x$ in $I = I_H \cup J$. By symmetry we prove finally that $|w_\ell| \leq c\varepsilon^{-3} \beta_\pm(r)$ in $I_x$. \hfill \Box

The next three lemmas present bounds for the third or higher order terms.

**Lemma 5.3.** For any $x \in I_x^\circ \setminus \{0\}$ it holds that

(5.19) \hspace{1cm} |w_{xxx}| \leq c\varepsilon^{-4} r^{-1} \beta_\pm(r),

(5.20) \hspace{1cm} |w_{xxt}| \leq c\varepsilon^{-4} \beta_\pm(r).
Proof. We consider \( x \in (0, \ell/2 + \varepsilon) \) so \( \varepsilon^2 w_{xx} = f'(\phi)w \). By differentiating the previous in \( x \) and using (5.7) and the \( |\phi_x| \) estimate, or by differentiating in \( \ell \) and using (5.7) and the \( |\phi_{\ell}| \) estimate we get the following

\[
|w_{xxx}| \leq c\varepsilon^{-2} \left[ |f'(\phi)||w_x| + |f''(\phi)||\phi_x||w| \right] \\
\leq c\varepsilon^{-2} \left[ c\varepsilon^{-2} r^{-1}\beta + c\varepsilon^{-1}\varepsilon^{-1}\beta \right] \leq c\varepsilon^{-4} r^{-1}\beta,
\]

and

\[
|w_{x\ell}| \leq c\varepsilon^{-2} \left[ |f'(\phi)||w_{\ell}| + |f''(\phi)||\phi_{\ell}||w| \right] \\
\leq c\varepsilon^{-2} \left[ c\varepsilon^{-2} \beta + c\varepsilon^{-1}\varepsilon^{-1}\beta \right] \leq c\varepsilon^{-4} \beta,
\]

for \( \beta = \beta_- \). Therefore, we obtain the results in \( I_s^o \setminus \{0\} \). \( \square \)

Lemma 5.4. For any \( x \in I_s \setminus \{0\} \) it holds that

\[
(5.21) \quad |w_{x\ell\ell}| \leq c\varepsilon^{-4} r^{-1}\beta_{\pm}(r).
\]

Proof. We consider \( x \in (0, \ell/2 + \varepsilon) \), write \( w_{x\ell\ell}(\ell/2) - w_{x\ell\ell}(x) = \int_x^{\ell/2} w_{x\ell\ell}(s) ds \) and get

\[
(5.22) \quad |w_{x\ell\ell}(x)| \leq |w_{x\ell\ell}(\ell/2)| + \int_x^{\ell/2} |w_{x\ell\ell}(s)| ds.
\]

We use the definition of \( w \) given in (5.10), set \( p = \varepsilon^{-1} \ell^2 \alpha' \), and remind that \( A = \int_{\ell/2}^{\ell} \frac{ds}{\phi_{\ell}^2} \). We take first the \( x \) derivative and then the \( \ell \ell \) derivative to obtain

\[
A_{x\ell}(\ell/2) = \frac{1}{2} \phi_x(\ell/2)^{-2}, \quad A_{\ell\ell}(\ell/2) = \phi_x(\ell/2)^{-3} \phi_{\ell\ell}(\ell/2) + \phi_{x\ell}(\ell/2) \phi_x(\ell/2)^{-3}.
\]

We also denote that \( \ell/2 \in J \), so by the estimates of Lemma 5.2 we obtain \( |\phi_{x\ell}(\ell/2)| \leq c\varepsilon^{-2} \) and \( |\phi_x(\ell/2)|^{-1} \leq c\varepsilon^{-4} \). Thus, as in Lemma 5.2 for general \( x \in J \), we get that \( c\varepsilon^{-4} |A_x(\ell/2)| + |A_{\ell\ell}(\ell/2)| \leq c\varepsilon^{-4} \).

In addition using the last estimate of (5.12) we obtain that \( |p_{\ell\ell}(\ell/2)| \leq c\varepsilon^{-5} \alpha \). Further, we use that \( \ell/2 \in J \), so by the proof of Lemma 5.2 we have that \( |\phi_x(\ell/2)| \leq c\varepsilon^{-2} \), while \( |\phi_{x\ell}(\ell/2)| \leq c\varepsilon^{-3} \), and therefore, we obtain finally

\[
(5.23) \quad |w_{x\ell\ell}(\ell/2)| \leq c\varepsilon^{-4} \alpha,
\]

for \( \alpha = \alpha_- \).

Since \( \varepsilon^2 w_{xx} = f'(\phi)w \) in \((0, \ell/2 + \varepsilon)\), then taking the \( \ell \ell \) derivative we arrive at

\[
(5.24) \quad |w_{x\ell\ell\ell}(x)| \leq c\varepsilon^{-2} \left[ |\phi_x|^2 |w| + |\phi_{\ell}| |w_{\ell}| + |w_{\ell\ell}| \right] \leq c\varepsilon^{-5} \beta
\]

for \( \beta = \beta_- \). Here, we used the estimates of the proof of Lemma 5.2 i.e. that \( |\phi_{\ell}| \leq c\varepsilon^{-1} \), the first and third estimate of (5.7), the fact that \( |w| \leq c\varepsilon^{-1} \beta \) while \( |w_{\ell}| \leq c\varepsilon^{-2} \beta \), and the result of Lemma 5.2 i.e. that \( |w_{\ell\ell}| \leq c\varepsilon^{-3} \beta \).

Since \( x \in (0, \ell/2) \) then using that \( r = \varepsilon/\ell \), we get that \( |x - \ell/2| \leq c(\ell/2 + \varepsilon) \leq c\varepsilon r^{-1} \), and therefore, (5.22), (5.23) and (5.24) give

\[
|w_{x\ell\ell}(x)| \leq c\varepsilon^{-4} r^{-1}\beta, \quad x \in (0, \ell/2 + \varepsilon).
\]

By symmetry the analogous result holds for any \( x \in [-\ell/2 - \varepsilon, 0) \). \( \square \)

Analogously the next lemma follows:
Lemma 5.5. For any \( x \in I_* - \{0\} \) it holds that
\[
|w_{xxxx}| + |w_{xx}e| + |w_{x\ell}| \leq c\varepsilon^{-5}r^{-1}\beta_\pm(r).
\]

According to the definition of \( E^\xi_i \), in order to estimate \( E^\xi_i, E^\xi_{ij} \) and \( E^\xi_{ijk} \) we need first the next result.

Lemma 5.6. For any \( i, j, k \) it follows that
\[
|Q_i| \leq c\varepsilon^{-3}\beta, \\
|Q_{ij}| \leq c\varepsilon^{-4}r^{-1}\beta, \\
|Q_{ijk}| \leq c\varepsilon^{-5}r^{-1}\beta.
\]

Proof. We remind that
\[
u^h_j(x) = \begin{cases}
\chi^{-1}w^j & \text{for } x \in I_{j-1} \\
(1 - \chi^j)(-\phi^j_x + w^j) + \chi^j(-\phi^{j+1}_x - w^{j+1}) + \chi^j_\pm(\phi^j - \phi^{j+1}) & \text{for } x \in I_j \\
-(1 - \chi^{j+1})w^{j+1} & \text{for } x \in I_{j+1},
\end{cases}
\]

Consider \( x = 0, 1 \) (i.e. in the first and last set of the support). Using the estimates of \( |w|, |w_{xx}| \) we arrive at
\[
|\tilde{u}^h_j| \leq c\varepsilon^{-1}\beta \text{ and thus } |\tilde{w}_j| \leq c\varepsilon^{-1}\beta, \\
|\tilde{u}^h_{jxx}| \leq c\varepsilon^{-3}\beta \text{ and thus } |\tilde{w}_{jxx}| \leq c\varepsilon^{-3}\beta.
\]
The estimates of \( |w_x|, |w_\ell| \) and of \( |w_{xxx}|, |w_{\ell xx}| \) respectively, now give
\[
|\tilde{u}^h_{jxx}| \leq c\varepsilon^{-2}r^{-1}\beta \text{ and thus } |\tilde{w}_{jxx}| \leq c\varepsilon^{-2}r^{-1}\beta, \\
|\tilde{u}^h_{jxx}| \leq c\varepsilon^{-4}r^{-1}\beta \text{ and thus } |\tilde{w}_{jxx}| \leq c\varepsilon^{-4}r^{-1}\beta.
\]

Finally, using the estimates of \( |w_{xx}|, |w_{x\ell}|, |w_{\ell\ell}| \) and of \( |w_{xxxx}|, |w_{xx\ell}|, |w_{x\ell\ell}| \) respectively we obtain
\[
|\tilde{u}^h_{jxx}| \leq c\varepsilon^{-3}r^{-1}\beta \text{ and thus } |\tilde{w}_{jxx}| \leq c\varepsilon^{-3}r^{-1}\beta, \\
|\tilde{u}^h_{jxx}| \leq c\varepsilon^{-5}r^{-1}\beta \text{ and thus } |\tilde{w}_{jxx}| \leq c\varepsilon^{-5}r^{-1}\beta.
\]

Remind also that
\[
\tilde{w}_j := \tilde{u}^h_j(x) + \tilde{u}^h_{j+1}(x), \\
Q_j(x) := (-\frac{1}{6}x^3 + \frac{1}{2}x^2 - \frac{1}{3}x)\tilde{w}_{jxx}(0) + \frac{1}{6}(x^3 - x)\tilde{w}_{jxx}(1) + x\tilde{w}_j(1), \ j = 1, \ldots, N,
\]
thus, the definition of \( Q_j \) combined with the above estimates on \( \tilde{w}_j \) give the result. \( \square \)

Remark 5.7. By [21] p. 557-556, the next estimates hold true
\[
\int_{-\ell/2}^0 \phi_x(x, \ell, -1)^2 + \int_{0}^{\ell/2} \phi_x(x, \ell, +1)^2 \leq c^{-1}S_\infty + E(r),
\]
where \( |E| \leq c\varepsilon^{-1}\beta \) and \( S_\infty = \int_{-1}^1 \sqrt{2F(u)}du \), and
\[
\int_{-\ell/2}^{\ell/2} |\phi_x|^2 dx \leq 2,
\]
and
\[
\int_{-\ell/2}^{\ell/2} |\phi_{xx}|^2 dx \leq c\varepsilon^{-3}.
\]
In addition, there exists constant \( c > 0 \) such that for \( x \in [h_j - \varepsilon, h_j + \varepsilon], j = 0, \ldots, N + 1 \) we have
\[
|\phi^j(x) - \phi^{j+1}(x)| \leq c|a^j - a^{j+1}|,
\]
Proof. For any $\xi_j$, $\xi_{j+1} < r_0$ with $r_0$ small (cf. [7]).

Now, we are able to compute bounds for the terms $\tilde{u}^h$, $u^h$ which are presented in the next theorem.

**Theorem 5.8.** For any $i, j, k$ it holds that

\[
\|\tilde{u}^h_i\|_{\infty} \leq \mathcal{O}(1) + \mathcal{O}(\|w\|_{\infty}),
\]

\[
\|\tilde{u}^h_i\| \leq ce^{-\frac{1}{2}}(1 + S_{\infty}^{1/2} + \max(r_ja^j, r_{j+1}a^{j+1})^{1/2}) + c\|w_x\| + c\|w_\ell\|,
\]

\[
\|u^h_j\| \leq \mathcal{O}(\varepsilon^{-1}),
\]

\[
\|u^h_j\|_{\infty} \leq \mathcal{O}(\varepsilon^{-2}),
\]

\[
\|u^h_j\| \leq \mathcal{O}(\varepsilon^{-\frac{1}{2}}).
\]

**Proof.** We use the definition of $u^h_j$ and get by (5.28) that

\[
\|\tilde{u}^h_i\| \leq c \int_0^1 |\phi_x| dx + c\|w\|_{\infty} \leq c + c\|w\|_{\infty}.
\]

Also, it follows that $\|u^h_j\| = \mathcal{O}(|\phi_x|) = \mathcal{O}(\varepsilon^{-1})$, thus

\[
\|u^h_j\|_{\infty} \leq \mathcal{O}(\varepsilon^{-1}).
\]

By (7) p. 38 it holds that

\[
u^h_j(x) = -u^h_x(x) + (1 - \chi^j)w^j - \chi^j w^{j+1} \quad x \in I_j,
\]

so using the above and (5.33) we obtain

\[
u^h_j(x) = \begin{cases} 
\mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j-1} \\
-u^h_x(x) + \mathcal{O}(w_x + w_\ell) & \text{for } x \in I_j \\
\mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j+1},
\end{cases}
\]

and therefore we arrive at

\[
\tilde{u}^h_j(x) = \int_0^x u^h_j(y) dy = \begin{cases} 
\mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j-1} \\
\mathcal{O}(u^h_x + w_x + w_\ell) & \text{for } x \in I_j \\
\mathcal{O}(w_x + w_\ell) & \text{for } x \in I_{j+1}.
\end{cases}
\]

By (21) (cf. p. 563, relation (8.6)) it holds that

\[
\|u^h_j\| \leq \varepsilon^{-1/2}(S_{\infty}^{1/2} + \max(r_ja^j, r_{j+1}a^{j+1})^{1/2}).
\]

Using the above estimate we obtain

\[
\|\tilde{u}^h_j\| \leq ce^{-\frac{1}{2}}(1 + S_{\infty}^{1/2} + \max(r_ja^j, r_{j+1}a^{j+1})^{1/2}) + c\|w_x\| + c\|w_\ell\|.
\]

Observe now that

\[
\tilde{u}^h_{j,ik}(x) = \int_0^x u^h_{j,ik}(y) dy = \begin{cases} 
\mathcal{O}(w_{xx} + w_{xt} + w_\ell) & \text{for } x \in I_{j-1} \\
\mathcal{O}(u^h_{x,ik} + w_{xx} + w_{xt} + w_\ell) & \text{for } x \in I_j \\
\mathcal{O}(w_{xx} + w_{xt} + w_\ell) & \text{for } x \in I_{j+1}.
\end{cases}
\]

In addition, since $u^h_j = -u^h_x + (1 - \chi^j)w^j - \chi^j w^{j+1}$ in $I_j$, then we obtain that

\[
\|u^h_{x,i}\| \leq \|u^h_{x,ik}\| + c\|w_x\|.
\]
The argument of [21] p. 562 of Lemma 8.3 applied for $u^h$ in $I_j$ and the support of $|\phi^{i}_x - \phi^{i+1}_x|$ combined with (5.31) and (5.28), since
\[ u^h_x = O(|\phi_x|) + O(|\phi_x - \phi^{i+1}_x|), \]
now gives that
\[ (5.36) \quad \|u^h_x\| \leq \|\phi_x\| + \sqrt{O(\varepsilon^{-2}\varepsilon)} \leq c\varepsilon^{-\frac{1}{2}}. \]
Analogously, differentiating two times in $x$ the function $u^h$ and using the estimate (5.32) and (5.29) and the support of $|\phi^{i}_xx - \phi^{i+1}_xx|$ we get
\[ u^h_{xx} = O(|\phi_{xx}|) + O(|\phi^{i}_xx - \phi^{i+1}_xx|), \]
and thus
\[ (5.37) \quad \|u^h_{xx}\| \leq \|\phi_{xx}\| + \sqrt{O(\varepsilon^{-4}\varepsilon)} \leq c\varepsilon^{-3/2}. \]
So, it follows that
\[ (5.38) \quad \|u^h_{xx}\| \leq c\varepsilon^{-3/2} + c\|w_x\|. \]
The previous estimates give finally
\[ \|\tilde{u}^h_{ijk}\| \leq c\varepsilon^{-3/2} + c\|w_x\| + c\|w_x\| + c\|w_{xx}\| + c\|w_{x\ell}\| + c\|w_{\ell\ell}\|. \]
Using again (5.34) we obtain
\[ (5.39) \quad |u^h_{ij}| \leq O(u^h_{ij}) = O(u^h_{xx}) = O(\varepsilon^{-2}), \]
therefore
\[ \|u^h_{ij}\|_{\infty} \leq O(\varepsilon^{-2}). \]
Further, by (5.34) and (5.36) it follows that
\[ (5.40) \quad \|u^h_{ij}\| \leq O(\|u^h_{ij}\|) = O(\varepsilon^{-\frac{1}{2}}), \]
and thus
\[ \|u^h_{ij}\| \leq O(\varepsilon^{-\frac{1}{2}}). \]

Using now the estimate $|\tilde{u}^h_{ijk}| \leq O(1) + O(\|w\|)$ combined with the implicit function result for change of variables we get that
\[ (5.41) \quad |\tilde{u}^h_{ij}| \leq (O(1) + O(\|w\|))(O(1) + O(\varepsilon^{-1}\beta)), \]
while the second derivative in $\xi$ variables gives
\[ \tilde{u}^h_{\xi} \leq |\tilde{u}^h_{\xi,j}||O(1) + O(\varepsilon^{-1}\beta)|^2 + |\tilde{u}^h_{\xi,j}||O(1) + O(\varepsilon^{-1}\beta)| + |\tilde{u}^h_{ij}|O(\varepsilon). \]
So, the next lemma follows.

**Lemma 5.9.** For any $j, k$ it holds
\[ (5.42) \quad \|\tilde{u}^h_{\xi,j}\| \leq (O(1) + O(\|w\|))(O(1) + O(\varepsilon^{-1}\beta)), \]
\[ (5.43) \quad \|\tilde{u}^h_{\xi,j}\| \leq [O(1) + O(\varepsilon^{-2}\beta^2) + O(\varepsilon^{-1}\beta)]|O(w_x + w_t) + \varepsilon^{-1/2} + \varepsilon^{-1/2}A| + O(\varepsilon)O(1) + O(\|w\|), \]
for $A := S^{1/2}_\infty + \max_j (r_j a_j, r_{j+1} a_j^i)^{1/2}$. 

The following theorem gives the final estimates concerning the term $E^e_\xi$ in the $L^2$ norm.
Theorem 5.10. For any $i, j, k$ the next inequalities hold:

$$\|E^\xi_{ji}\| \leq 4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta),$$  
(5.41)

$$\|E^\xi_{ij}\| \leq \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4}\beta),$$  
(5.42)

$$\|E^\xi_{ijk}\| \leq \mathcal{O}(\varepsilon^{-3/2}) + \mathcal{O}(\varepsilon^{-5}\beta).$$  
(5.43)

Proof. Using that $\|E^\xi_{ij}\| \leq \|\tilde{w}_j\| + \|Q_j\|$, the estimate of $\|\tilde{w}_j\|$ presented in [8] (cf. p. 186, relation (4.24)) and Lemma 5.6 we obtain (5.41). Also, observe that

$$E^\xi_{ji} = \tilde{w}_{ji} + O(Q_{ji}) + O(Q_{jix}) = O(w_x + w_t) + \int_0^x (-u^h_{xj}) dy + O(Q_{ji}) + O(Q_{jix})$$

$$\leq O(w_x + w_t) + O(u^h_i) + O(Q_{ji}),$$
so, by (5.35) and Lemma 5.6 we get

$$\|E^\xi_{ji}\| \leq \mathcal{O}(\varepsilon^{-1/2}) + \mathcal{O}(\varepsilon^{-4}\beta).$$

Further, using (5.34) we obtain

$$E^\xi_{jk} = \tilde{w}_{ijk} + O(Q_{jik}) + O(Q_{jikx}) = O(w_{xx} + w_{\ell t} + w_{xt}) + \int_0^x (-u^h_{xjk}) dy + O(Q_{jik}) + O(Q_{jikx})$$

$$\leq O(w_{xx} + w_{\ell t} + w_{xt}) + O(u^h_i) + O(Q_{jik}),$$
so, by (5.37) and Lemma 5.6 we get

$$\|E^\xi_{ijk}\| \leq \mathcal{O}(\varepsilon^{-3/2}) + \mathcal{O}(\varepsilon^{-5}\beta).$$

Remark 5.11. We denote that the estimate of $\|E^\xi_{ij}\|$ presented in the previous Theorem coincides in the main order term with the estimate that was used but not presented analytically in [8].

Using the results of the previous analysis we derive finally by Cauchy-Schwarz inequality all the desired estimates involving the higher order derivatives which are presented at the next main theorem of this section.

Theorem 5.12. The next inequalities hold for any $i, j, l, k$:

$$\langle \tilde{u}^\xi_{ki}, E^\xi_{ij} \rangle \leq \mathcal{O}(\varepsilon^{-1/2}) \left[ 4\ell_{i+1} + \mathcal{O}(\varepsilon^{-3}\beta) \right],$$  
(5.44)

$$\langle \tilde{u}^\xi_{k}, E^\xi_{ij} \rangle \leq \mathcal{O}(\varepsilon^{-1/2} + \varepsilon^{-4}\beta),$$  
(5.45)

and

$$\langle \tilde{v}, E^\xi_{ijk} \rangle \leq \mathcal{O}(\varepsilon^{-3/2} + \varepsilon^{-5}\beta)\|\tilde{v}\|.$$  
(5.46)

The last term to be analyzed is $\langle L^r\tilde{v}, \tilde{u}^\xi_{kl} \rangle$. Therefore, we prove the following main result.

Theorem 5.13. For any $k, l$, it holds that

$$\langle L^r\tilde{v}, \tilde{u}^\xi_{kl} \rangle \leq \varepsilon^{-5}\beta(r)^2 \left( \mathcal{O}(1) + \varepsilon^{-2}\beta(r)^2 \right)\|\tilde{v}\|.$$  
(5.47)

Proof. We denote that

$$\langle L^r\tilde{v}, \tilde{u}^\xi_{kl} \rangle = -\langle \tilde{v}, \partial_x \partial_{\xi^i} \partial_{\xi^j} \mathcal{L}^b(u^\xi) \rangle,$$
where $\mathcal{L}^b(\phi) := \varepsilon^2 \phi_{xx} - f(\phi)$. As in [7] (cf. p. 452-453) we write for $x \in [h_j - \varepsilon, h_j + \varepsilon], j = 1, 2, \ldots, N + 1$

$$\mathcal{L}^b(u^b) = f_1 + f_2 + G,$$  
(5.48)
Denote that in the above computation the worst term is

\[ |\chi_{x}^3(\phi^{j+1} - \phi^j)|, \quad f_2 := 2\varepsilon^2 \chi_{x}^j(\phi_{x}^{j+1} - \phi_{x}^j), \]

\[ G := (\phi^{j+1} - \phi^j)^2 \left\{ (1 - \chi^j) \int_0^{\chi^j} s f''(\theta) ds + \chi^j \int_{\chi^j}^1 (1 - s) f''(\theta) ds \right\}, \]

with \( \theta = \theta(s) := (1 - s) \phi'(x) + s \phi'(x) \). For other \( x \), we use \( L_b^b(u^b) = 0 \).

In Lemma 5.2 of [7] at p. 454, after differentiating \( f_1, f_2, G \) in \( h_j \) is derived that

\[ \left| \frac{\partial}{\partial h_j} L_b^b u^b \right| \leq c\varepsilon^{-2} \beta(r). \]

Applying the analogous computation, i.e. differentiating in \( h_j, h_i \), we may derive

\[ (5.49) \quad \left| \frac{\partial^2}{\partial h_j \partial h_i} L_b^b u^b \right| \leq c\varepsilon^{-3} \beta(r). \]

Denote that in the above computation the worst term is \( |\phi_{xxx}^j(x) - \phi_{xxx}^{j+1}(x)| \). But \( \varepsilon^2 \phi_{xxx} = f'(\phi) \phi_x \), where \( f(\phi) = \phi^3 - \phi \) and \( f'(\phi) = 3\phi^2 - 1 \), so using the estimates of \( \phi, \phi_x \) and the results for the differences presented at p. 453 of [7], we get

\[ |\phi_{xxx}^j(x) - \phi_{xxx}^{j+1}(x)| = \varepsilon^{-2} |f'(\phi)\phi_{x}^j(x) - f'(\phi)\phi_{x}^{j+1}(x)| \]

\[ = \varepsilon^{-2} |f'(\phi)\phi_{x}^j(x) - f'(\phi)\phi_{x}^{j+1}(x) - f'(\phi)\phi_{x}^{j+1}(x) + f'(\phi)\phi_{x}^{j+1}(x)| \]

\[ \leq \varepsilon^{-2} |f'(\phi)||\phi_{x}^j(x) - \phi_{x}^{j+1}(x)| + \varepsilon^{-2} |\phi_{x}^{j+1}(x)| |f'(\phi) - f'(\phi)| \]

\[ \leq \varepsilon^{-2} |\phi_{x}^j(x) - \phi_{x}^{j+1}(x)| + \varepsilon^{-2} |f'(\phi) - f'(\phi)| \]

\[ \leq \varepsilon^{-3} |a^j - a^{j+1}| + \varepsilon^{-3} |3\phi_{x}^j(x)^2 - 1 - 3\phi_{x}^{j+1}(x)^2 + 1| \]

\[ \leq \varepsilon^{-3} |a^j - a^{j+1}| + \varepsilon^{-3} |\phi_{x}^j(x) + \phi_{x}^{j+1}| |\phi_{x}^j(x) - \phi_{x}^{j+1}| \]

\[ \leq \varepsilon^{-3} |a^j - a^{j+1}| + \varepsilon^{-3} |a^j - a^{j+1}| \]

\[ \leq \varepsilon^{-3} |a^j - a^{j+1}|. \]

Again as in [7] (cf. p. 456), by using that \( \varepsilon^2 w_{xx} = f'(\phi(x))w \) and differentiating it in \( x \), we may obtain that

\[ (5.50) \quad \left| \frac{\partial^2}{\partial h_j \partial h_i} L_b^b u^b \right| \leq c\varepsilon^{-5} \beta(r). \]

Returning now in \( \xi \) variables, since the second derivative appears, then by use of the formula (5.1) in (5.50) and since (cf. [7] p. 454) it holds that

\[ (5.51) \quad \left| \frac{\partial}{\partial h_j} L_b^b u^b \right| \leq c\varepsilon^{-4} \beta(r), \]

we obtain finally

\[ (5.52) \quad \left| \frac{\partial^2}{\partial \xi_k \partial \xi_l} L_b^b u^b \right| \leq \varepsilon^{-5} \beta(r) \left\{ (\mathcal{O}(1) + \varepsilon^{-1} \beta(r))^2 + (\mathcal{O}(1) + \varepsilon^{-1} \beta(r)) \right\} + \varepsilon^{-4} \beta(r) \mathcal{O}(\varepsilon) \]

\[ \leq \varepsilon^{-5} \beta(r) \left( \mathcal{O}(1) + \varepsilon^{-2} \beta(r)^2 \right). \]

So, the result follows. \( \square \)

**Acknowledgment.** The third author is supported by a Marie Curie International Reintegration Grant within the 7th European Community Framework Programme, MIRG-CT-2007-200526, and partially supported by the FP7-REGPOT-2009-1 project “Archimedes Center for Modeling, Analysis and Computation”.

---

30 ANTONOPOLOU, BLÖMKER, KARALI

(5.51)
References


