# Finite element approximations for a linear fourth-order parabolic SPDE in two and three dimensions with additive space-time white noise

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# Finite element approximations for a linear fourth-order parabolic SPDE in two and three space dimensions with additive space-time white noise☆

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**Abstract**

We consider an initial- and Dirichlet boundary- value problem for a linear fourth-order stochastic parabolic equation, in two or three space dimensions, forced by an additive space-time white noise. Discretizing the space-time white noise a modeling error is introduced and a regularized fourth-order linear stochastic parabolic problem is obtained. Fully-discrete approximations to the solution of the regularized problem are constructed by using, for discretization in space, a standard Galerkin finite element method based on $H^2-$piecewise polynomials, and, for time-stepping, the Backward Euler method. We derive strong a priori estimates for the modeling error and for the approximation error to the solution of the regularized problem.

*Keywords:* finite element method, space-time white noise, Backward Euler time-stepping, fully-discrete approximations, a priori error estimates, fourth order parabolic equation, two and three space dimensions
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## 1. Introduction

### *1.1. Formulation of the problem*

Let $d = 2$ or 3, $T > 0$, $D = (0,1)^d \subset \mathbb{R}^d$ and $(\Omega, \mathcal{F}, P)$ be a complete probability space. Then we consider an initial- and Dirichlet boundary- value problem for a fourth-order linear stochastic parabolic equation formulated, typically, as follows: find a stochastic function $u : [0,T] \times \overline{D} \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t u + \Delta^2 u &= \dot{W}(t,x) \quad \forall\, (t,x) \in (0,T] \times D, \\ \Delta^m u(t,\cdot)\big|_{\partial D} &= 0 \quad \forall\, t \in (0,T], \quad m = 0,1, \\ u(0,x) &= 0 \quad \forall\, x \in D, \end{aligned} \tag{1.1}$$

a.s. in $\Omega$, where $\dot{W}$ denotes a space-time white noise on $[0,T] \times D$ (see, e.g.,[27], [16]). The stochastic partial differential equation in (1.1) is the linear diffusive part of the stochastic Cahn-Hilliard equation (cf. [5], [10]) which was introduced for the investigation of phase separation in spinodal decomposition (see, e.g., [6], [17], [12]).

The mild solution of the problem above (cf. [5], [10]), known as 'stochastic convolution', is given by

$$u(t,x) = \int_0^t \int_D G(t-s;x,y)\, dW(s,y). \tag{1.2}$$

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Here, $G(t;x,y)$ is the space-time Green kernel of the corresponding deterministic parabolic problem: find a deterministic function $w:[0,T]\times\overline{D}\to\mathbb{R}$ such that

$$\begin{aligned} \partial_t w+\Delta^2 w=0 \quad &\forall\,(t,x)\in(0,T]\times D,\\ \Delta^m w(t,\cdot)\big|_{\partial D}=0 \quad &\forall\,t\in(0,T],\quad m=0,1,\\ w(0,x)=w_0(x) \quad &\forall\,x\in D, \end{aligned} \tag{1.3}$$

where $w_0$ is a deterministic initial condition. In particular, we have

$$w(t,x)=\int_D G(t;x,y)\,w_0(y)\,dy \quad \forall\,(t,x)\in(0,T]\times\overline{D}$$

and

$$G(t;x,y)=\sum_{\alpha\in\mathbb{N}^d} e^{-\lambda_\alpha^2 t}\,\varepsilon_\alpha(x)\,\varepsilon_\alpha(y) \quad \forall\,(t,x,y)\in(0,T]\times\overline{D}\times\overline{D}, \tag{1.4}$$

where $\lambda_\alpha:=\pi^2\,|\alpha|^2_{\mathbb{N}^d}$, $|\alpha|_{\mathbb{N}^d}:=\left(\sum_{i=1}^d\alpha_i^2\right)^{\frac12}$ and $\varepsilon_\alpha(z):=2^{\frac{d}{2}}\prod_{i=1}^d\sin(\alpha_i\,\pi\,z_i)$ for all $z\in\overline{D}$ and $\alpha\in\mathbb{N}^d$.

*1.2. The regularized problem*

Extending the approach proposed in [1] for a second order one-dimensional linear stochastic parabolic equation with additive space-time white noise, we construct below an approximate initial and boundary value problem:

For $N_\star,\,J_\star\in\mathbb{N}$, define the mesh-lengths $\Delta t:=\frac{T}{N_\star}$, $\Delta x:=\frac{1}{J_\star}$, and the nodes $t_n:=n\,\Delta t$ for $n=0,\dots,N_\star$ and $x_j:=j\,\Delta x$ for $j=0,\dots,J_\star$. Then, we define the sets $\mathcal{N}_\star:=\{1,\dots,N_\star\}$, $\mathcal{J}_\star:=\{1,\dots,J_\star\}$, $T_n:=(t_{n-1},t_n)$ for $n\in\mathcal{N}_\star$, $D_j:=(x_{j-1},x_j)$ for $j\in\mathcal{J}_\star$, $D_\mu:=\prod_{i=1}^d D_{\mu_i}$ for $\mu\in\mathcal{J}_\star^d$, and $S_{n,\mu}:=T_n\times D_\mu$ for $n\in\mathcal{N}_\star$ and $\mu\in\mathcal{J}_\star^d$. Next, consider the fourth-order linear stochastic parabolic problem:

$$\begin{aligned} \partial_t\widehat{u}+\Delta^2\widehat{u}=\widehat{W} \quad &\text{in}\ \ (0,T]\times D,\\ \Delta^m\widehat{u}(t,\cdot)\big|_{\partial D}=0 \quad &\forall\,t\in(0,T],\quad m=0,1,\\ \widehat{u}(0,x)=0 \quad &\forall\,x\in D, \end{aligned} \tag{1.5}$$

a.e. in $\Omega$, where

$$\widehat{W}(t,x):=\tfrac{1}{\Delta t\,(\Delta x)^d}\sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}_\star}\mathcal{X}_{S_{n,\mu}}(t,x)\,R^{n,\mu} \quad \forall\,(t,x)\in[0,T]\times\overline{D},$$

$$R^{n,\mu}:=\int_{S_{n,\mu}}1\,dW \quad ,\forall\,n\in\mathcal{N}_\star,\ \ \forall\,\mu\in\mathcal{J}_\star^d,$$

and $\mathcal{X}_S$ is the index function of $S\subset[0,T]\times\overline{D}$.

The solution of the problem (1.5), according to the standard theory for parabolic problems (see, e.g, [22]), has the integral representation

$$\widehat{u}(t,x)=\int_0^t\!\!\int_D G(t-s;x,y)\,\widehat{W}(s,y)\,dsdy \quad \forall\,(t,x)\in[0,T]\times\overline{D}. \tag{1.6}$$

**Remark 1.** The properties of the stochastic integral (see, e.g., [27]), yield that $R^{n,\mu}\sim\mathcal{N}(0,\Delta t(\Delta x)^d)$ for all $(n,\mu)\in\mathcal{N}_\star\times\mathcal{J}_\star^d$. Also, we observe that $\mathbb{E}[R^{n,\mu}\,R^{n',\mu'}]=0$ for $(n,\mu)\neq(n',\mu')$. Thus, the random variables $(R^{n,\mu})_{(n,\mu)\in\mathcal{N}_\star\times\mathcal{J}_\star^d}$ are independent.

*1.3. The numerical approximations*

In order to construct fully-discrete approximations to $\widehat{u}$, we let $M \in \mathbb{N}$, $(\tau_m)_{m=0}^{M}$ be the nodes of a uniform partition of $[0,T]$ with stepsize $\Delta\tau$, i.e. $\tau_m := m\,\Delta\tau$ for $m = 0,\dots,M$, and define $\Delta_m := (\tau_{m-1},\tau_m)$ for $m = 1,\dots,M$. Also, we let $M_h \subset H_0^1(D) \cap H^2(D)$ be a finite element space consisting of functions which are piecewise polynomials over a partition of $D$ in triangles or rectangulars with maximum diameter $h$, and define a discrete biharmonic operator $B_h : M_h \to M_h$ by

$$\int_D B_h\varphi\ \chi\ dx = \int_D \Delta\varphi\ \Delta\chi\ dx, \quad \forall\,\varphi,\chi \in M_h,$$

and the usual $L^2(D)-$projection operator $P_h : L^2(D) \to M_h$ by

$$\int_D P_h f\ \chi\ dx = \int_D f\ \chi\ dx, \quad \forall\,\chi \in M_h, \quad \forall\,f \in L^2(D).$$

The approximations to $\widehat{u}$ we consider follow by employing the Backward Euler finite element method which begins by setting

$$\widehat{U}_h^0 := 0, \tag{1.7}$$

and, then for $m = 1,\dots,M$, finds $\widehat{U}_h^m \in M_h$ such that

$$\widehat{U}_h^m - \widehat{U}_h^{m-1} + \Delta\tau\, B_h \widehat{U}_h^m = \int_{\Delta_m} P_h \widehat{W}\ ds. \tag{1.8}$$

*1.4. Main results of the paper*

In the rest of the paper we investigate the convergence of the fully discrete approximations to the solution $\widehat{u}$ of (1.5) to the mild solution $u$ of (1.1). That error of approximating $u$ splits in two parts: the *modeling error* which is the error of approximating $u$ by $\widehat{u}$, and the *numerical approximation error* which is the error of approximating $\widehat{u}$ by the numerical method defined in (1.7)–(1.8).

An $L_t^\infty(L_P^2(L_x^2))$ estimate of the modeling error is achieved, in Theorem 5, by obtaining the bound

$$\max_{t\in[0,T]} \left\{ \int_\Omega \left( \int_D |u(t,x) - \widehat{u}(t,x)|^2\ dx \right) dP \right\}^{\frac{1}{2}} \le C \left[ \epsilon^{-\frac{1}{2}}\ \Delta x^{\frac{4-d}{2}-\epsilon} + \Delta t^{\frac{4-d}{8}} \right], \quad \forall\,\epsilon \in (0, \tfrac{4-d}{2}],$$

without imposing conditions on $\Delta t$ and $\Delta x$ as happens in [1] and [2]. For the numerical approximation error, we derive, in Theorem 11, the following discrete in time $L_t^\infty(L_P^2(L_x^2))$ estimate:

$$\max_{0\le m\le M} \left\{ \int_\Omega \left( \int_D \left|\widehat{U}_h^m(x) - \widehat{u}(\tau_m,x)\right|^2\ dx \right) dP \right\}^{\frac{1}{2}} \le C \left[ \epsilon_1^{-\frac{1}{2}}\ \Delta\tau^{\frac{4-d}{8}-\epsilon_1} + \epsilon_2^{-\frac{1}{2}}\ h^{\nu_\star-\epsilon_2} \right], \tag{1.9}$$

for $\epsilon_1 \in (0, \frac{4-d}{8}]$ and $\epsilon_2 \in (0, \nu_\star]$, where $\nu_\star = \nu_\star(r,d)$ is given in (5.26) and depends on the space dimension $d$ and a parameter $r \in \{2,3,4\}$ which is related to the approximation properties of the finite element spaces $M_h$ (see (2.19)). To get the estimate (1.9), first we introduce the Backward-Euler time-discrete approximations of $\widehat{u}$ and analyze their convergence in the discrete in time $L_t^\infty(L_P^2(L_x^2))$ norm above (see Theorem 7); then, we derive an estimate for the error of approximating the Backward-Euler time-discrete approximations of $\widehat{u}$ by the Backward-Euler fully-discrete approximation of $\widehat{u}$ (see Proposition 10). This procedure allows us to estimate separately the space and the time discretization error in constrast to the technique used in [26] and [2] for second order problems.

For approximation methods for fourth-order stochastic parabolic problems driven by a space-time white noise, we refer the reader: to [4] which considers a finite difference method for the stochastic Cahn-Hilliard equation, and to [24], [14] and [15] which consider time-stepping methods for a wide family of evolution problems that includes (1.1), while the finite element method is not among the space-discretization techniques considered in [14] and [15]. Our previous paper [20] analyzes Backward Euler finite element approximations for the 1D space dimensional case where the space regularity of the solution is higher and thus a different regularized problem is proposed as a basis for developing the numerical method. We also refer to [21] for the analysis of a Backward Euler finite element method for problem (1.1), where the biharmonic operator $\Delta^2$ is discretized by $\Delta_h^2$, $\Delta_h$ being the discrete Laplacian operator (see, e.g., [25]). In the present paper we use the discrete operator $B_h$ for the discretization of the biharmonic operator which is different from $\Delta_h^2$. Also, we refer the reader to [8], [1], [18], [26], [28] and [2] for the analysis of the finite element method for second order stochastic parabolic problems.

We close the section by an overview of the paper. Section 2 introduces notation, and recalls or prove several results often used in the paper. Section 3 is dedicated to the estimation of the modeling error. Section 4 defines the Backward Euler time-discrete approximations of $\widehat{u}$ and analyzes its convergence. Section 5 contains the error analysis for the Backward Euler fully-discrete approximations of $\widehat{u}$.

## 2. Notation and preliminaries

### 2.1. Function spaces and operators

We denote by $L^2(D)$ the space of the Lebesgue measurable functions which are square integrable on $D$ with respect to Lebesgue's measure $dx$, provided with the standard norm $\|g\|_{0,D} := \{\int_D |g(x)|^2\, dx\}^{\frac{1}{2}}$ for $g \in L^2(D)$. The standard inner product in $L^2(D)$ that produces the norm $\|\cdot\|_{0,D}$ is written as $(\cdot,\cdot)_{0,D}$, i.e., $(g_1,g_2)_{0,D} := \int_D g_1(x)g_2(x)\, dx$ for $g_1,\, g_2 \in L^2(D)$. For $s \in \mathbb{N}_0$, $H^s(D)$ will be the Sobolev space of functions having generalized derivatives up to order $s$ in the space $L^2(D)$, and by $\|\cdot\|_{s,D}$ its usual norm, i.e. $\|g\|_{s,D} := \left\{\sum_{\alpha\in\mathbb{N}_0^d,\, |\alpha|_{\mathbb{N}^d}\le s} \|\partial_x^\alpha g\|_{0,D}^2\right\}^{\frac{1}{2}}$ for $g \in H^s(D)$. Also, by $H_0^1(D)$ we denote the subspace of $H^1(D)$ consisting of functions which vanish at the boundary $\partial D$ of $D$ in the sense of trace. We note that in $H_0^1(D)$ the, well-known, Poincaré-Friedrichs inequality holds, i.e.,

$$\|g\|_{0,D} \le C_{PF}\, \|\nabla g\|_{0,D} \quad \forall\, g \in H_0^1(D), \tag{2.1}$$

where $\|\nabla v\|_{0,D} := \left(\sum_{\alpha\in\mathbb{N}_0^d,\, |\alpha|_{\mathbb{N}^d}=1} \|\partial_x^\alpha v\|_{0,D}^2\right)^{\frac{1}{2}}$ for $v \in H^1(D)$.

The sequence of pairs $\{(\lambda_\alpha, \varepsilon_\alpha)\}_{\alpha\in\mathbb{N}^d}$ is a solution to the eigenvalue/eigenfunction problem: find nonzero $\varphi \in H^2(D)\cap H_0^1(D)$ and $\sigma \in \mathbb{R}$ such that $-\Delta\varphi = \sigma\,\varphi$ in $D$. Since $(\varepsilon_\alpha)_{\alpha\in\mathbb{N}^d}$ is a complete $(\cdot,\cdot)_{0,D}$–orthonormal system in $L^2(D)$, for $s \in \mathbb{R}$, a subspace $\dot{\mathbf{H}}^s(D)$ of $L^2(D)$ (see [25]) is defined by

$$\dot{\mathbf{H}}^s(D) := \left\{ v \in L^2(D) : \quad \sum_{\alpha\in\mathbb{N}^d} \lambda_\alpha^s\, (v,\varepsilon_\alpha)_{0,D}^2 < \infty \right\}$$

and provided with the norm $\|v\|_{\dot{\mathbf{H}}^s} := \left(\sum_{\alpha\in\mathbb{N}^d} \lambda_\alpha^s\, (v,\varepsilon_\alpha)_{0,D}^2\right)^{\frac{1}{2}} \quad \forall\, v \in \dot{\mathbf{H}}^s(D)$. Let $m \in \mathbb{N}_0$. It is well-known (see [25]) that

$$\dot{\mathbf{H}}^m(D) = \left\{ v \in H^m(D) : \quad \Delta^i v\,|_{\partial D} = 0 \quad \text{if} \;\; 0 \le i < \tfrac{m}{2} \right\} \tag{2.2}$$

and there exist constants $C_{m,A}$ and $C_{m,B}$ such that

$$C_{m,A}\, \|v\|_{m,D} \le \|v\|_{\dot{\mathbf{H}}^m} \le C_{m,B}\, \|v\|_{m,D} \quad \forall\, v \in \dot{\mathbf{H}}^m(D). \tag{2.3}$$

Also, we define on $L^2(D)$ the negative norm $\|\cdot\|_{-m,D}$ by

$$\|v\|_{-m,D} := \sup\left\{ \tfrac{(v,\varphi)_{0,D}}{\|\varphi\|_{m,D}} : \quad \varphi \in \dot{\mathbf{H}}^m(D) \;\; \text{and} \;\; \varphi \ne 0 \right\} \quad \forall\, v \in L^2(D),$$

for which, using (2.3), it is easy to conclude that there exists a constant $C_{-m} > 0$ such that

$$\|v\|_{-m,D} \le C_{-m}\, \|v\|_{\dot{\mathbf{H}}^{-m}} \quad \forall\, v \in L^2(D). \tag{2.4}$$

Let $\mathbb{L}_2 = (L^2(D), (\cdot,\cdot)_{0,D})$ and $\mathcal{L}(\mathbb{L}_2)$ be the space of linear, bounded operators from $\mathbb{L}_2$ to $\mathbb{L}_2$. We say that, an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is *Hilbert-Schmidt*, when $\|\Gamma\|_{\rm HS} := \left(\sum_{k=1}^{\infty} \|\Gamma\varepsilon_k\|_{0,D}^2\right)^{\frac{1}{2}} < +\infty$, where $\|\Gamma\|_{\rm HS}$ is the so called Hilbert-Schmidt norm of $\Gamma$. We note that the quantity $\|\Gamma\|_{\rm HS}$ does not change when we replace $(\varepsilon_k)_{k=1}^{\infty}$ by another complete orthonormal system of $\mathbb{L}_2$. It is well known (see, e.g., [11]) that an operator $\Gamma \in \mathcal{L}(\mathbb{L}_2)$ is Hilbert-Schmidt iff there exists a measurable function $g : D \times D \to \mathbb{R}$ such that $\Gamma[v](\cdot) = \int_D g(\cdot, y)\, v(y)\, dy$ for $v \in L^2(D)$, and then, it holds that

$$\|\Gamma\|_{\rm HS} = \left(\int_D\int_D g^2(x,y)\, dxdy\right)^{\frac{1}{2}}. \tag{2.5}$$

Let $\mathcal{L}_{\rm HS}(\mathbb{L}_2)$ be the set of Hilbert Schmidt operators of $\mathcal{L}(\mathbb{L}^2)$ and $\Phi : [0,T] \to \mathcal{L}_{\rm HS}(\mathbb{L}_2)$. Also, for a random variable $X$, let $\mathbb{E}[X]$ be its expected value, i.e., $\mathbb{E}[X] := \int_\Omega X\, dP$. Then, the Itô isometry property for stochastic integrals, which we will use often in the paper, reads

$$\mathbb{E}\left[\left\| \int_0^T \Phi\, dW \right\|_{0,D}^2\right] = \int_0^T \|\Phi(t)\|_{\rm HS}^2\, dt. \tag{2.6}$$

For later use, we introduce the projection operator $\widehat{\Pi} : L^2((0,T)\times D) \to L^2((0,T)\times D)$ defined by

$$\widehat{\Pi}(g;\cdot)\,\big|_{S_{n,\mu}} := \tfrac{1}{\Delta t\, \Delta x^d} \int_{S_{n,\mu}} g(t,x)\, dtdx, \quad \forall\, n \in \mathcal{N}_\star, \quad \forall\, \mu \in \mathcal{J}_\star^d, \tag{2.7}$$

for $g \in L^2((0,T)\times D)$, which obviously satisfies that

$$\left(\int_0^T\int_D (\widehat{\Pi} g)^2\, dxdt\right)^{\frac{1}{2}} \leq \left(\int_0^T\int_D g^2\, dxdt\right)^{\frac{1}{2}} \quad \forall\, g \in L^2((0,T)\times D). \tag{2.8}$$

and has the following property:

**Lemma 1.** *For $g \in L^2((0,T)\times D)$, it holds that*

$$\int_0^T\int_D \widehat{\Pi}(g;s,y)\, dW(s,y) = \int_0^T\int_D \widehat{W}(t,x)\, g(t,x)\, dtdx. \tag{2.9}$$

*Proof.* To obtain (2.9) we work, using (2.7) and the properties of $W$, as follows:

$$\begin{aligned}
\int_0^T\int_D \widehat{\Pi}(g;s,y)\, dW(s,y) =& \tfrac{1}{\Delta t\,(\Delta x)^d} \sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}_\star^d} \left(\int_{S_{n,\mu}} g\; dtdx\right)\left(\int_0^T\int_D \mathcal{X}_{S_{n,\mu}}(s,y)\, dW(s,y)\right)\\
=& \tfrac{1}{\Delta t\,(\Delta x)^d} \sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}_\star^d} \left(\int_{S_{n,\mu}} g(t,x)\; dtdx\right) R^{n,\mu}\\
=& \tfrac{1}{\Delta t\,(\Delta x)^d} \sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}_\star^d} \int_0^T\int_D g(t,x)\, \mathcal{X}_{S_{n,\mu}}(t,x)\, R^{n,\mu}\, dtdx\\
=& \int_0^T\int_D g(t,x)\, \widehat{W}(t,x)\, dtdx.
\end{aligned}$$

□

We close this section, by stating some asymptotic bounds for series that will often appear in the rest of the paper and for a proof of them we refer the reader to [19].

**Lemma 2.** *Let $d \in \{1,2,3\}$ and $c_\star > 0$. Then, there exists a constant $C > 0$ that depends on $c_\star$ and $d$, such that*

$$\sum_{\alpha\in\mathbb{N}^d} |\alpha|_{\mathbb{N}^d}^{-(d+c_\star\epsilon)} \leq C\,\epsilon^{-1} \quad \forall\, \epsilon \in (0,2]. \tag{2.10}$$

**Lemma 3.** *Let $d \in \{2,3\}$ and $\delta > 0$. Then there exists a constant $C > 0$ which is independent of $\delta$, such that*

$$\sum_{\alpha \in \mathbb{N}^d} \tfrac{1 - e^{-\lambda_\alpha^2 \delta}}{\lambda_\alpha^2} \le C \;\; p_d(\delta^{\frac{1}{4}}) \;\; \delta^{\frac{4-d}{4}}, \tag{2.11}$$

*where $p_d(s) := 1 + \sum_{i=1}^{d} s^i$.*

*2.2. Linear elliptic and parabolic operators*

For given $f \in L^2(D)$ let $v_{\scriptscriptstyle E} \in H^2(D) \cap H_0^1(D)$ be the solution of the boundary value problem

$$\Delta v_{\scriptscriptstyle E} = f \quad \text{in} \;\; D, \tag{2.12}$$

and $T_{\scriptscriptstyle E} : L^2(D) \to H^2(D) \cap H_0^1(D)$ be its solution operator, i.e. $T_{\scriptscriptstyle E} f := v_{\scriptscriptstyle E}$, which has the property

$$\|T_{\scriptscriptstyle E} f\|_{m,D} \le C_{E,m} \, \|f\|_{m-2,D}, \quad \forall \, f \in H^{\max\{0,m-2\}}(D), \;\; \forall \, m \in \mathbb{N}_0. \tag{2.13}$$

Also, for $f \in L^2(D)$ let $v_{\scriptscriptstyle B} \in H^4(D)$ be the solution of the following biharmonic boundary value problem

$$\begin{aligned} \Delta^2 v_{\scriptscriptstyle B} = f \quad \text{in} \;\; D, \\ \Delta^m v_{\scriptscriptstyle B}\big|_{\partial D} = 0, \quad m = 0, 1, \end{aligned} \tag{2.14}$$

and $T_{\scriptscriptstyle B} : L^2(D) \to \dot{\mathbf{H}}^4(D)$ be the solution operator of (2.14), i.e. $T_{\scriptscriptstyle B} f := v_{\scriptscriptstyle B}$, which satisfies

$$\|T_{\scriptscriptstyle B} f\|_{m,D} \le C_{B,m} \, \|f\|_{m-4,D}, \quad \forall \, f \in H^{\max\{0,m-4\}}(D), \;\; \forall \, m \in \mathbb{N}_0. \tag{2.15}$$

Due to the type of boundary conditions of (2.14), we conclude that

$$T_{\scriptscriptstyle B} f = T_{\scriptscriptstyle E}^2 f, \quad \forall \, f \in L^2(D), \tag{2.16}$$

which, easily, yields

$$(T_{\scriptscriptstyle B} v_1, v_2)_{0,D} = (T_{\scriptscriptstyle E} v_1, T_{\scriptscriptstyle E} v_2)_{0,D} \quad \forall \, v_1, v_2 \in L^2(D). \tag{2.17}$$

Letting $(\mathcal{S}(t) w_0)_{t \in [0,T]}$ be the standard semigroup notation for the solution $w$ of (1.3), we can easily establish the following property (see, e.g., [25], [23]): for $\ell \in \mathbb{N}_0$, $\beta$, $p \in \mathbb{R}_0^+$ and $q \in [0, p + 4\ell]$ there exists a constant $C > 0$ such that:

$$\int_{t_a}^{t_b} (t - t_a)^\beta \left\| \partial_t^\ell \mathcal{S}(t) w_0 \right\|_{\dot{\mathbf{H}}^p}^2 dt \le C \, \|w_0\|_{\dot{\mathbf{H}}^{p+4\ell-2\beta-2}}^2 \quad \forall \, t_b > t_a \ge 0, \;\; \forall \, w_0 \in \dot{\mathbf{H}}^{p+4\ell-2\beta-2}(D). \tag{2.18}$$

*2.3. Discrete spaces and operators*

For $r \in \{2, 3, 4\}$, we consider a finite element space $M_h \subset H_0^1(D) \cap H^2(D)$ consisting of functions which are piecewise polynomials over a partition of $D$ in triangles or rectangles with maximum mesh-length $h$. We assume that the space $M_h$ has the following approximation property

$$\inf_{\chi \in M_h} \|v - \chi\|_{2,D} \le C \, h^{r-1} \, \|v\|_{r+1,D} \quad \forall \, v \in H^{r+1}(D) \cap H_0^1(D), \tag{2.19}$$

which covers several classes of $H^2$ finite element spaces, for example the tensor products of $C^1$ splines, the Argyris triangle elements, the Hsieh-Clough-Tocher triangle elements and the Bell triangle (cf. [7], [3]).

A finite element approximation $v_{\scriptscriptstyle B,h} \in M_h$ of the solution $v_{\scriptscriptstyle B}$ of (2.14) is defined by the requirement

$$B_h v_{\scriptscriptstyle B,h} = P_h f. \tag{2.20}$$

Then, we denote by $T_{B,h} : L^2(D) \to M_h$ the solution operator of (2.20), i.e. $T_{B,h}f := v_{B,h} = B_h^{-1} P_h f$ for $f \in L^2(D)$, which satisfies that

$$(T_{B,h}f, g)_{0,D} = (\Delta T_{B,h}f, \Delta T_{B,h}g)_{0,D} = (f, T_{B,h}g)_{0,D} \quad \forall\, f, g \in L^2(D), \tag{2.21}$$

Also, using (2.20), (2.14) and (2.15) we conclude that

$$\begin{aligned} \|\Delta T_{B,h}f\|_{0,D} \leq& \|\Delta T_B f\|_{0,D} \\ \leq& C\,\|f\|_{-2,D} \qquad \forall\, f \in L^2(D). \end{aligned} \tag{2.22}$$

Applying the standard theory of the finite element method (see, e.g., [7], [3]) and using (2.15), we get

$$\|\Delta(T_B f - T_{B,h}f)\|_{0,D} \leq C\,h^{r-1}\,\|f\|_{r-3,D}, \quad \forall\, f \in H^{\max\{r-3,0\}}(D), \tag{2.23}$$

while error estimates in the $L^2(D)$ norm are obtained in the proposition below.

**Proposition 4.** *Let $r \in \{2, 3, 4\}$. Then, it holds that:*

$$\|T_B f - T_{B,h}f\|_{0,D} \leq C \begin{cases} h^5\,\|f\|_{1,D}, & r = 4 \\ h^4\,\|f\|_{0,D}, & r = 3, \\ h^2\,\|f\|_{-1,D}, & r = 2, \end{cases} \qquad \forall\, f \in H^{\max\{r-3,0\}}(D). \tag{2.24}$$

*Proof.* Let $f \in H^{\max\{0,r-3\}}(D)$ and $e = T_B f - T_{B,h}f$. Also, we define a bilinear form $\gamma : H^2(D) \times H^2(D) \to \mathbb{R}$ by $\gamma(v_1, v_2) := (\Delta v_1, \Delta v_2)_{0,D}$ for $v_1$, $v_2 \in H^2(D)$. Now, let $w_A$, $w_B \in \dot{\mathbf{H}}^4(D)$ be defined by $T_B \Delta e = w_A$ and $T_B e = w_B$. Then, using Galerkin orthogonality, we have:

$$\begin{aligned} \|\nabla e\|_{0,D}^2 =& -\gamma(w_A, e)_{0,D} \\ \leq& \|\Delta e\|_{0,D} \inf_{\chi \in M_h} \|w_A - \chi\|_{2,D} \end{aligned} \tag{2.25}$$

and

$$\begin{aligned} \|e\|_{0,D}^2 =& \gamma(w_B, e)_{0,D} \\ \leq& \|\Delta e\|_{0,D} \inf_{\chi \in M_h} \|w_B - \chi\|_{2,D}. \end{aligned} \tag{2.26}$$

*Case 1*: Let $r \in \{2, 3\}$. Then, using (2.26), (2.23), (2.19) and (2.22), we obtain

$$\begin{aligned} \|e\|_{0,D}^2 \leq& C\,h^{r-1}\,\|f\|_{r-3,D}\,h^{r-1}\,\|w_B\|_{r+1,D} \\ \leq& C\,h^{2(r-1)}\,\|f\|_{r-3,D}\,\|e\|_{r-3,D} \end{aligned}$$

which, obviously, yields (2.24).

*Case 2*: Let $r = 4$. Then, combining, (2.26), (2.19), (2.15) and (2.1), we get

$$\begin{aligned} \|e\|_{0,D}^2 \leq& C\,\|\Delta e\|_{0,D}\,h^3\,\|T_B e\|_{5,D} \\ \leq& C\,\|\Delta e\|_{0,D}\,h^3\,\|e\|_{1,D} \\ \leq& C\,\|\Delta e\|_{0,D}\,h^3\,\|\nabla e\|_{0,D}. \end{aligned} \tag{2.27}$$

Also, we observe that (2.25) and (2.15) yield

$$\begin{aligned} \|\nabla e\|_{0,D} \leq& \|\Delta e\|_{0,D}^{\frac{1}{2}}\,\|\Delta(T_B \Delta e)\|_{0,D}^{\frac{1}{2}} \\ \leq& \|\Delta e\|_{0,D}^{\frac{1}{2}}\,\|e\|_{0,D}^{\frac{1}{2}}. \end{aligned} \tag{2.28}$$

Now, we combine (2.27), (2.28) and (2.23) to have

$$\begin{aligned} \|e\|_{0,D}^{\frac{3}{2}} \leq& C\,h^3\,\|\Delta e\|_{0,D}^{\frac{3}{2}} \\ \leq& C\,h^{\frac{15}{2}}\,\|f\|_{1,D}^{\frac{3}{2}}, \end{aligned}$$

which obviously leads to (2.24) for $r = 4$. □

**Remark 2.** In the estimate (2.24) we observe that the order of convergence is equal to $r+1$ except in the case $r=2$. Note that this is not in contradiction to the results in [13] where only the case $r\geq 3$ is considered.

## 3. An estimate for the modeling error

Here, we derive an $L^\infty_t(L^2_P(L^2_x))$ bound for the modeling error $u-\widehat{u}$, in terms of $\Delta t$ and $\Delta x$.

**Theorem 5.** *Let $u$ and $\widehat{u}$ be defined, respectively, by* (1.2) *and* (1.6)*. Then, there exists a real constant $C>0$, independent of $T$, $\Delta t$ and $\Delta x$, such that*

$$\max_{[0,T]}\left\{\mathbb{E}\left[\|u-\widehat{u}\|^2_{0,D}\right]\right\}^{\frac{1}{2}}\leq C\,\left[\,(p_d(\Delta t^{\frac{1}{4}}))^{\frac{1}{2}}\;\;\Delta t^{\frac{4-d}{8}}+\epsilon^{-\frac{1}{2}}\;\;\Delta x^{\frac{4-d}{2}-\epsilon}\;\right]\quad\forall\,\epsilon\in(0,\tfrac{4-d}{2}],\tag{3.1}$$

*where $p_d$ is the polynomial defined in Lemma 3.*

*Proof.* Using (1.2) and (1.6), we conclude that

$$u(t,x)-\widehat{u}(t,x)=\int_0^T\!\!\int_D\left[\mathcal{X}_{(0,t)}(s)\,G(t-s;x,y)-\widetilde{G}(t,x;s,y)\right]dW(s,y)\quad\forall\,(t,x)\in[0,T]\times\overline{D},\tag{3.2}$$

where $\widetilde{G}:(0,T)\times D\to L^2((0,T)\times D)$ given by

$$\widetilde{G}(t,x;\cdot)\Big|_{S_{n,\mu}}\equiv\tfrac{1}{\Delta t\,(\Delta x)^d}\int_{S_{n,\mu}}\mathcal{X}_{(0,t)}(s')\,G(t-s';x,y')\;ds'dy'\tag{3.3}$$

for $n\in\mathcal{N}_\star$ and $\mu\in\mathcal{J}^d_\star$.

Let $\Theta:=\left(\mathbb{E}\left[\|u-\widehat{u}\|^2_{0,D}\right]\right)^{\frac{1}{2}}$ and $t\in(0,T]$. Using (3.2), the Itô isometry (2.6) and (2.5), we obtain

$$\Theta^2(t)=\int_0^T\left(\int_D\!\int_D\left[\mathcal{X}_{(0,t)}(s)\,G(t-s;x,y)-\widetilde{G}(t,x;s,y)\right]^2dxdy\right)\,ds$$

from which, using (3.3), follows that

$$\Theta(t)=\tfrac{1}{\Delta t\,(\Delta x)^d}\left\{\sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}^d_\star}\int_D\left\{\int_{S_{n,\mu}}\left[\int_{S_{n,\mu}}\left[\mathcal{X}_{(0,t)}(s)\,G(t-s;x,y)-\mathcal{X}_{(0,t)}(s')\,G(t-s';x,y')\right]ds'dy'\right]^2dsdy\right\}dx\right\}^{\frac{1}{2}}.$$

Now, we introduce the splitting

$$\Theta(t)\leq\;\Theta_A(t)+\Theta_B(t),\tag{3.4}$$

where

$$\Theta_A(t):=\tfrac{1}{\Delta t\,(\Delta x)^d}\left\{\sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}^d_\star}\int_D\left\{\int_{S_{n,\mu}}\left[\int_{S_{n,\mu}}\mathcal{X}_{(0,t)}(s)\left[G(t-s;x,y)-G(t-s;x,y')\right]ds'dy'\right]^2dsdy\right\}dx\right\}^{\frac{1}{2}}$$

and

$$\Theta_B(t)=\tfrac{1}{\Delta t\,(\Delta x)^d}\left\{\sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}^d_\star}\int_D\left\{\int_{S_{n,\mu}}\left[\int_{S_{n,\mu}}\left[\mathcal{X}_{(0,t)}(s)\,G(t-s;x,y')-\mathcal{X}_{(0,t)}(s')\,G(t-s';x,y')\right]ds'dy'\right]^2dsdy\right\}dx\right\}^{\frac{1}{2}}.$$

`Estimation of` $\Theta_{A}(t)$: Using (1.4) and the $(\cdot,\cdot)_{0,D}-$orthogonality of $(\varepsilon_\alpha)_{\alpha\in\mathbb{N}^d}$, we have

$$\begin{aligned}
\Theta_{A}^2(t) =&\ \tfrac{1}{(\Delta x)^{2d}} \sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}_\star^d}\int_D\left\{\int_{S_{n,\mu}}\left[\int_{D_\mu}\mathcal{X}_{(0,t)}(s)\left[G(t-s;x,y)-G(t-s;x,y')\right]dy'\right]^2 dsdy\right\}dx\\
=&\ \tfrac{1}{(\Delta x)^{2d}} \sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}_\star^d}\left\{\int_{S_{n,\mu}}\left[\sum_{\alpha\in\mathbb{N}^d}\mathcal{X}_{(0,t)}(s)\,e^{-2\lambda_\alpha^2(t-s)}\left(\int_{D_\mu}(\varepsilon_\alpha(y)-\varepsilon_\alpha(y'))\,dy'\right)^2\right]dsdy\right\}\\
=&\ \tfrac{1}{(\Delta x)^{2d}} \sum_{\alpha\in\mathbb{N}^d}\left\{\sum_{n\in\mathcal{N}_\star}\int_{T_n}\mathcal{X}_{(0,t)}(s)\,e^{-2\lambda_\alpha^2(t-s)}\,ds\right\}\left\{\sum_{\mu\in\mathcal{J}_\star^d}\int_{D_\mu}\left(\int_{D_\mu}(\varepsilon_\alpha(y)-\varepsilon_\alpha(y'))\,dy'\right)^2dy\right\},\\
=&\ \tfrac{1}{(\Delta x)^{2d}} \sum_{\alpha\in\mathbb{N}^d}\left\{\int_0^t e^{-2\lambda_\alpha^2(t-s)}\,ds\right\}\left\{\sum_{\mu\in\mathcal{J}_\star^d}\int_{D_\mu}\left(\int_{D_\mu}(\varepsilon_\alpha(y)-\varepsilon_\alpha(y'))\,dy'\right)^2dy\right\},
\end{aligned}$$

from which, using the Cauchy-Schwarz inequality, follows that

$$\Theta_{A}^2(t)\le\sum_{\alpha\in\mathbb{N}^d}\left(\int_0^t e^{-2\lambda_\alpha^2(t-s)}\,ds\right)\left[\tfrac{1}{(\Delta x)^d}\sum_{\mu\in\mathcal{J}_\star^d}\int_{D_\mu\times D_\mu}\left|\varepsilon_\alpha(y)-\varepsilon_\alpha(y')\right|^2dy'dy\right]. \tag{3.5}$$

Observing that $\int_0^t e^{-2\lambda_\alpha^2(t-s)}\,ds\le\frac{1}{2}\,\lambda_\alpha^{-2}$ for $\alpha\in\mathbb{N}^d$, and that

$$\begin{aligned}
\sup_{y,y'\in D_\mu}\left|\varepsilon_\alpha(y)-\varepsilon_\alpha(y')\right| &\le 2^{\frac{d}{2}+1}\,\min\left\{1,\tfrac{\pi}{2}\,d^{\frac12}\,\Delta x\,|\alpha|_{\mathbb{N}^d}\right\}\\
&\le 2^{\frac{d}{2}+1-\gamma}\,\pi^\gamma\,d^{\frac{\gamma}{2}}\,\Delta x^\gamma\,|\alpha|_{\mathbb{N}^d}^\gamma,\quad \forall\,\gamma\in[0,1],\quad\forall\,\alpha\in\mathbb{N}^d,\quad\forall\,\mu\in\mathcal{J}_\star^d,
\end{aligned}$$

(3.5) yields

$$\Theta_{A}^2(t)\le\ 2^{d+1-2\gamma}\,d^\gamma\,\pi^{2\gamma-4}\,(\Delta x)^{2\gamma}\sum_{\alpha\in\mathbb{N}^d}\tfrac{1}{|\alpha|_{\mathbb{N}^d}^{2(2-\gamma)}}. \tag{3.6}$$

The series in (3.6) converges when $2(2-\gamma)>d$ or equivalently $\gamma<\frac{4-d}{2}$. Thus, combining (3.6) and (2.10), we, finally, conclude that

$$\Theta_{A}(t)\le\ C\,\epsilon^{-\frac12}\,\Delta x^{\frac{4-d}{2}-\epsilon}\quad\forall\,\epsilon\in\left(0,\tfrac{4-d}{2}\right]. \tag{3.7}$$

`Estimation of` $\Theta_{B}(t)$: For $t\in(0,T]$, let $\widehat{N}(t):=\min\left\{\ell\in\mathbb{N}:\ \ 1\le\ell\le N_\star\ \text{ and }\ t\le t_\ell\right\}$ and

$$\widehat{T}_n(t):=T_n\cap(0,t)=\begin{cases}T_n, & \text{if }\ n<\widehat{N}(t)\\ (t_{\widehat{N}(t)-1},t), & \text{if }\ n=\widehat{N}(t)\end{cases},\qquad n=1,\dots,\widehat{N}(t).$$

Now, we use (1.4) and the $(\cdot,\cdot)_{0,D}-$orthogonality of $(\varepsilon_\alpha)_{\alpha\in\mathbb{N}^d}$ as follows

$$\begin{aligned}
\Theta_{B}^2(t)=&\ \tfrac{(\Delta x)^d}{(\Delta t\,(\Delta x)^d)^2}\sum_{n\in\mathcal{N}_\star}\sum_{\mu\in\mathcal{J}_\star^d}\int_D\left\{\int_{T_n}\left[\int_{S_{n,\mu}}\Big[\mathcal{X}_{(0,t)}(s)\,G(t-s;x,y')\right.\right.\\
&\qquad\left.\left.-\ \mathcal{X}_{(0,t)}(s')\,G(t-s';x,y')\Big]\,ds'dy'\right]^2ds\right\}dx\\
=&\ \tfrac{(\Delta x)^d}{(\Delta t\,(\Delta x)^d)^2}\sum_{\alpha\in\mathbb{N}^d}\left[\sum_{\mu\in\mathcal{J}_\star^d}\left(\int_{D_\mu}\varepsilon_\alpha(y')\,dy'\right)^2\right]\left[\sum_{n=1}^{\widehat{N}(t)}\int_{T_n}\left(\int_{T_n}\Big(\mathcal{X}_{(0,t)}(s)\,e^{-\lambda_\alpha^2(t-s)}\right.\right.\\
&\qquad\left.\left.-\ \mathcal{X}_{(0,t)}(s')\,e^{-\lambda_\alpha^2(t-s')}\Big)\,ds'\right)^2ds\right]
\end{aligned}$$

which yields that

$$\Theta_{B}^2(t) \le \; 2^d \sum_{\alpha\in\mathbb{N}^d} \left( \tfrac{1}{(\Delta t)^2} \sum_{n=1}^{\widehat{N}(t)} \Psi_n^{\alpha}(t) \right), \tag{3.8}$$

where

$$\Psi_n^{\alpha}(t) := \int_{T_n} \Big( \int_{T_n} \Big( \mathcal{X}_{(0,t)}(s)\, e^{-\lambda_\alpha^2(t-s)} - \mathcal{X}_{(0,t)}(s')\, e^{-\lambda_\alpha^2(t-s')} \Big)\; ds' \Big)^2 \, ds.$$

Let $\alpha\in\mathbb{N}^d$ and $n\in\{1,\dots,\widehat{N}(t)-1\}$. Then, we have

$$\begin{aligned}
\Psi_n^{\alpha}(t) &= \int_{T_n} \Big( \int_{T_n}\int_{s}^{s'} \lambda_\alpha^2\, e^{-\lambda_\alpha^2(t-\tau)}\, d\tau ds' \Big)^2 \, ds \\
&\le \int_{T_n} \Big( \int_{T_n}\int_{t_{n-1}}^{\max\{s',s\}} \lambda_\alpha^2\, e^{-\lambda_\alpha^2(t-\tau)}\, d\tau ds' \Big)^2 \, ds \\
&\le 2\int_{T_n} \Big( \int_{T_n}\int_{t_{n-1}}^{s'} \lambda_\alpha^2\, e^{-\lambda_\alpha^2(t-\tau)}\, d\tau ds' \Big)^2 \, ds + 2\int_{T_n} \Big( \int_{T_n}\int_{t_{n-1}}^{s} \lambda_\alpha^2\, e^{-\lambda_\alpha^2(t-\tau)}\, d\tau\, ds' \Big)^2 \, ds \\
&\le 2\,\Delta t\Big( \int_{T_n}\int_{t_{n-1}}^{s'} \lambda_\alpha^2\, e^{-\lambda_\alpha^2(t-\tau)}\, d\tau ds' \Big)^2 + 2\,(\Delta t)^2 \int_{T_n} \Big( \int_{t_{n-1}}^{s} \lambda_\alpha^2\, e^{-\lambda_\alpha^2(t-\tau)}\, d\tau \Big)^2 \, ds,
\end{aligned}$$

from which, using the Cauchy-Schwarz inequality, follows that

$$\Psi_n^{\alpha}(t) \le \; 4\,(\Delta t)^2 \int_{T_n} \Big( \int_{t_{n-1}}^{s} \lambda_\alpha^2\, e^{-\lambda_\alpha^2(t-\tau)}\, d\tau \Big)^2 \, ds.$$

Now, observing that $\lambda_\alpha^2\, e^{\lambda_\alpha^2(\tau-t)} = \partial_\tau \left( e^{\lambda_\alpha^2(\tau-t)} \right)$, we obtain

$$\begin{aligned}
\Psi_n^{\alpha}(t) &\le \; 4\,(\Delta t)^2 \int_{T_n} \left( e^{-\lambda_\alpha^2(t-s)} - e^{-\lambda_\alpha^2(t-t_{n-1})} \right)^2 ds \\
&\le \; 4\,(\Delta t)^2 \left(1 - e^{-\lambda_\alpha^2\Delta t}\right)^2 \int_{T_n} e^{-2\lambda_\alpha^2(t-s)}\, ds \\
&\le \; 2\,(\Delta t)^2 \left(1 - e^{-\lambda_\alpha^2\Delta t}\right)^2 \tfrac{e^{-2\lambda_\alpha^2(t-t_n)} - e^{-2\lambda_\alpha^2(t-t_{n-1})}}{\lambda_\alpha^2}.
\end{aligned}$$

Thus, by summing with respect to $n$, we obtain

$$\tfrac{1}{(\Delta t)^2} \sum_{n=1}^{\widehat{N}(t)-1} \Psi_n^{\alpha}(t) \le \; 2\, \tfrac{(1-e^{-\lambda_\alpha^2\Delta t})^2}{\lambda_\alpha^2}. \tag{3.9}$$

Considering, now, the case $n=\widehat{N}(t)$, we have

$$\Psi_{\widehat{N}(t)}^{\alpha}(t) = \Psi_{A}^{\alpha}(t) + \Psi_{B}^{\alpha}(t) \tag{3.10}$$

with

$$\begin{aligned}
\Psi_{A}^{\alpha}(t) &:= \int_{t_{\widehat{N}(t)-1}}^{t} \left( \int_{t_{\widehat{N}(t)-1}}^{t} \int_{s'}^{s} \lambda_\alpha^2 e^{-\lambda_\alpha^2(t-\tau)}\, d\tau ds' + \int_{t}^{t_{\widehat{N}(t)}} e^{-\lambda_\alpha^2(t-s)}\, ds' \right)^2 ds \\
\Psi_{B}^{\alpha}(t) &:= \int_{t}^{t_{\widehat{N}(t)}} \left( \int_{t_{\widehat{N}(t)}-1}^{t} e^{-\lambda_\alpha^2(t-s')}\, ds' \right)^2 ds.
\end{aligned}$$

Then, we have

$$\begin{aligned}\Psi_{B}^{\alpha}(t) &\le \tfrac{\Delta t}{\lambda_{\alpha}^4}\left[1-e^{-\lambda_{\alpha}^2\left(t-t_{\widehat{N}(t)-1}\right)}\right]^2\\ &\le \tfrac{\Delta t}{\lambda_{\alpha}^4}\left(1-e^{-\lambda_{\alpha}^2\,\Delta t}\right)^2\end{aligned}$$

and

$$\begin{aligned}\Psi_{A}^{\alpha}(t) \le& \int_{t_{\widehat{N}(t)-1}}^{t}\left[\int_{t_{\widehat{N}(t)-1}}^{t}\int_{s'}^{s}\lambda_{\alpha}^2 e^{-\lambda_{\alpha}^2(t-\tau)}\,d\tau ds'+\Delta t\ \ e^{-\lambda_{\alpha}^2(t-s)}\right]^2 ds\\ \le&\ 2\int_{t_{\widehat{N}(t)-1}}^{t}\left[\int_{t_{\widehat{N}(t)-1}}^{t}\int_{s'}^{s}\lambda_{\alpha}^2 e^{-\lambda_{\alpha}^2(t-\tau)}\,d\tau ds'\right]^2 ds+\tfrac{(\Delta t)^2}{\lambda_{\alpha}^2}\left[1-e^{-2\lambda_{\alpha}^2\left(t-t_{\widehat{N}(t)-1}\right)}\right]\\ \le&\ 2\int_{t_{\widehat{N}(t)-1}}^{t}\left[\int_{t_{\widehat{N}(t)-1}}^{t}\int_{t_{\widehat{N}(t)-1}}^{\max\{s,s'\}}\lambda_{\alpha}^2 e^{-\lambda_{\alpha}^2(t-\tau)}\,d\tau ds'\right]^2 ds+\tfrac{(\Delta t)^2}{\lambda_{\alpha}^2}\left(1-e^{-2\lambda_{\alpha}^2\,\Delta t}\right)\\ \le&\ 8\,(\Delta t)^2\int_{t_{\widehat{N}(t)-1}}^{t}\left[\int_{t_{\widehat{N}(t)-1}}^{s}\lambda_{\alpha}^2 e^{-\lambda_{\alpha}^2(t-\tau)}\,d\tau\right]^2 ds+\tfrac{(\Delta t)^2}{\lambda_{\alpha}^2}\left(1-e^{-2\lambda_{\alpha}^2\,\Delta t}\right)\\ \le&\ 8\,(\Delta t)^2\int_{t_{\widehat{N}(t)-1}}^{t}\left[e^{-\lambda_{\alpha}^2(t-s)}-e^{-\lambda_{\alpha}^2(t-t_{\widehat{N}(t)-1})}\right]^2 ds+\tfrac{(\Delta t)^2}{\lambda_{\alpha}^2}\left(1-e^{-2\lambda_{\alpha}^2\,\Delta t}\right),\end{aligned}$$

which, along with (3.10), gives

$$\Psi_{\widehat{N}(t)}^{\alpha}\le 5\tfrac{(\Delta t)^2}{\lambda_{\alpha}^2}\left(1-e^{-2\lambda_{\alpha}^2\,\Delta t}\right)+\tfrac{\Delta t}{\lambda_{\alpha}^4}\left(1-e^{-\lambda_{\alpha}^2\Delta t}\right)^2.$$

Since the mean value theorem yields: $1-e^{-\lambda_{\alpha}^2\Delta t}\le\lambda_{\alpha}^2\,\Delta t$, the above inequality takes the form

$$\tfrac{1}{(\Delta t)^2}\,\Psi_{\widehat{N}(t)}^{\alpha}\le 6\,\tfrac{1-e^{-2\lambda_{\alpha}^2\,\Delta t}}{\lambda_{\alpha}^2}\,. \tag{3.11}$$

Combining (3.8), (3.9) and (3.11) we obtain

$$\Theta_{B}^2(t)\le 8\sum_{\alpha\in\mathbb{N}^d}\tfrac{1-e^{-2\lambda_{\alpha}^2\,\Delta t}}{\lambda_{\alpha}^2}\,. \tag{3.12}$$

Now, combine (3.12) and (2.11) to arrive at

$$\Theta_{B}(t)\le C\,(p_d(\Delta t^{\frac{1}{4}}))^{\frac{1}{2}}\ \Delta t^{\frac{4-d}{8}}. \tag{3.13}$$

The error bound (3.1) follows by observing that $\Theta(0)=0$ and combining the bounds (3.4), (3.7) and (3.13). □

## 4. Time-discrete approximations

The Backward Euler time-discrete approximations to the solution $\widehat{u}(\tau_m,\cdot)$ of the problem (1.5) are defined as follows: first, set

$$\widehat{U}^0:=0, \tag{4.1}$$

and then, for $m=1,\dots,M$, find $\widehat{U}^m\in\dot{\mathbf{H}}^4(D)$ such that

$$\widehat{U}^m-\widehat{U}^{m-1}+\Delta\tau\,\Delta^2\widehat{U}^m=\int_{\Delta_m}\widehat{W}\,ds\quad\text{a.s.}. \tag{4.2}$$

To develop an error estimate in a discrete in time $L^\infty_t(L^2_P(L^2_x))$ norm for the above time-discrete approximations, we need an error estimate in a discrete in time $L^2_t(L^2_x)$ norm for the Backward Euler time-discrete approximations, $(W^m)_{m=0}^M$, of the solution $w$ to the deterministic problem (1.3), specified by setting

$$W^0 := w_0, \tag{4.3}$$

and then, for $m=1,\dots,M$, by finding $W^m \in \dot{\mathbf{H}}^4(D)$ such that

$$W^m - W^{m-1} + \Delta\tau\,\Delta^2 W^m = 0. \tag{4.4}$$

**Proposition 6.** *Let $(W^m)_{m=0}^M$ be the Backward Euler time-discrete approximations of the solution $w$ of the problem* (1.3) *defined in* (4.3)–(4.4)*. If $w_0 \in \dot{\mathbf{H}}^2(D)$, then, there exists a constant $C>0$, independent of $T$ and $\Delta\tau$, such that*

$$\left(\sum_{m=1}^M \Delta\tau\,\|W^m - w(\tau_m,\cdot)\|^2_{0,D}\right)^{\frac12} \le C\,(\Delta\tau)^\theta\,\|w_0\|_{\dot{\mathbf{H}}^{4\theta-2}} \quad \forall\,\theta\in[0,1]. \tag{4.5}$$

*Proof.* It is analogous to the proof of Proposition 4.1 in [20], and thus is omitted. □

**Theorem 7.** *Let $\widehat{u}$ be the solution of* (1.5) *and $(\widehat{U}^m)_{m=0}^M$ be the Backward Euler time-discrete approximations specified in* (4.1)–(4.2)*. Then there exists a constant $C>0$, independent of $T$, $\Delta t$, $\Delta x$ and $\Delta\tau$, such that*

$$\max_{1\le m\le M}\left\{\mathbb{E}\left[\|\widehat{U}^m - \widehat{u}(\tau_m,\cdot)\|^2_{0,D}\right]\right\}^{\frac12} \le C\ \widetilde{\omega}(\Delta\tau,\epsilon)\ \Delta\tau^{\frac{4-d}{8}-\epsilon}, \quad \forall\,\epsilon\in\left(0,\tfrac{4-d}{8}\right], \tag{4.6}$$

*where $\widetilde{\omega}(\Delta\tau,\epsilon) := [\epsilon^{-\frac12} + (\Delta\tau)^\epsilon\,(p_d(\Delta\tau^{\frac14}))^{\frac12}]$ and $p_d$ is the polynomial defined in Lemma 3.*

*Proof.* Let $I: L^2(D)\to L^2(D)$ be the identity operator, $\Lambda: L^2(D)\to\dot{\mathbf{H}}^4(D)$ be the inverse elliptic operator $\Lambda := (I+\Delta\tau\,\Delta^2)^{-1}$ which has Green function $G_\Lambda(x,y) = \sum_{\alpha\in\mathbb{N}^d}\frac{\varepsilon_\alpha(x)\,\varepsilon_\alpha(y)}{1+\Delta\tau\lambda_\alpha^2}$, i.e. $\Lambda f(x) = \int_D G_\Lambda(x,y) f(y)\,dy$ for $x\in\overline{D}$ and $f\in L^2(D)$. Obviously, $G_\Lambda(x,y) = G_\Lambda(y,x)$ for $x,y\in D$, and $G\in L^2(D\times D)$. Also, for $m\in\mathbb{N}$, we denote by $G_{\Lambda,m}$ the Green function of $\Lambda^m$. Thus, from (4.2), using an induction argument, we conclude that $\widehat{U}^m = \sum_{j=1}^m\int_{\Delta_j}\Lambda^{m-j+1}\widehat{W}(\tau,\cdot)\,d\tau$ for $m=1,\dots,M$, which is written, equivalently, as follows:

$$\widehat{U}^m(x) = \int_0^{\tau_m}\!\!\int_D \widehat{\mathcal{K}}_m(\tau;x,y)\,\widehat{W}(\tau,y)\,dyd\tau \quad \forall\,x\in\overline{D}, \quad m=1,\dots,M, \tag{4.7}$$

where $\widehat{\mathcal{K}}_m(\tau;x,y) := \sum_{j=1}^m \mathcal{X}_{\Delta_j}(\tau)\,G_{\Lambda,m-j+1}(x,y) \quad \forall\,\tau\in[0,T], \ \ \forall\,x,y\in D.$

Let $m\in\{1,\dots,M\}$ and $\mathcal{E}^m := \mathbb{E}\big[\|\widehat{U}^m - \widehat{u}(\tau_m,\cdot)\|^2_{0,D}\big]$. First, we use (4.7), (1.6), (2.9), (2.6), (2.5) and (2.8), to obtain

$$\begin{aligned}
\mathcal{E}^m =\,&\mathbb{E}\Big[\int_D\Big(\int_0^T\!\!\int_D \mathcal{X}_{(0,\tau_m)}(\tau)\,\big[\widehat{\mathcal{K}}_m(\tau;x,y) - G(\tau_m-\tau;x,y)\big]\,\widehat{W}(\tau,y)\,dyd\tau\Big)^2 dx\Big]\\
\le\,&\int_0^{\tau_m}\left(\int_D\!\int_D \big[\widehat{\mathcal{K}}_m(\tau;x,y) - G(\tau_m-\tau;x,y)\big]^2 dydx\right)d\tau\\
\le\,&\sum_{\ell=1}^m\int_{\Delta_\ell}\left(\int_D\!\int_D \big[G_{\Lambda,m-\ell+1}(x,y) - G(\tau_m-\tau;x,y)\big]^2 dydx\right)d\tau.\\
\le\,&\sum_{\ell=1}^m\int_{\Delta_\ell}\|\Lambda^{m-\ell+1} - \mathcal{S}(\tau_m-\tau)\|^2_{\mathrm{HS}}\,d\tau\\
\le\,&\mathcal{B}^m_A + \mathcal{B}^m_B,
\end{aligned} \tag{4.8}$$

where

$$\mathcal{B}_A^m := 2\sum_{\ell=1}^{m}\int_{\Delta_\ell}\|\Lambda^{m-\ell+1}-\mathcal{S}(\tau_m-\tau_{\ell-1})\|_{\mathrm{HS}}^2\,d\tau,$$

$$\mathcal{B}_B^m := 2\sum_{\ell=1}^{m}\int_{\Delta_\ell}\|\mathcal{S}(\tau_m-\tau_{\ell-1})-\mathcal{S}(\tau_m-\tau)\|_{\mathrm{HS}}^2\,d\tau.$$

`Estimation of` $\mathcal{B}_A^m$: By the definition of the Hilbert-Schmidt norm, we have

$$\begin{aligned}
\mathcal{B}_A^m &\le 2\,\Delta\tau\sum_{\ell=1}^{m}\left(\sum_{\alpha\in\mathbb{N}^d}\|\Lambda^{m-\ell+1}\varepsilon_\alpha-\mathcal{S}(\tau_m-\tau_{\ell-1})\varepsilon_\alpha\|_{0,D}^2\right)\\
&\le 2\sum_{\alpha\in\mathbb{N}^d}\left(\sum_{\ell=1}^{m}\Delta\tau\,\|\Lambda^{m-\ell+1}\varepsilon_\alpha-\mathcal{S}(\tau_m-\tau_{\ell-1})\varepsilon_\alpha\|_{0,D}^2\right)\\
&\le 2\sum_{\alpha\in\mathbb{N}^d}\left(\sum_{\ell=1}^{m}\Delta\tau\,\|\Lambda^{\ell}\varepsilon_\alpha-\mathcal{S}(\tau_\ell)\varepsilon_\alpha\|_{0,D}^2\right).
\end{aligned}$$

Let $\theta\in[0,\frac{4-d}{8})$ and $\epsilon=\frac{4-d}{8}-\theta$. Using the deterministic error estimate (4.5) and (2.10), we obtain

$$\begin{aligned}
\mathcal{B}_A^m &\le C\,\Delta\tau^{2\theta}\sum_{\alpha\in\mathbb{N}^d}\|\varepsilon_\alpha\|_{\dot{\mathbf{H}}^{4\theta-2}}^2\\
&\le C\,\Delta\tau^{2\theta}\sum_{\alpha\in\mathbb{N}^d}\lambda_\alpha^{4\theta-2}\\
&\le C\,\Delta\tau^{2\theta}\sum_{\alpha\in\mathbb{N}^d}\tfrac{1}{|\alpha|_{\mathbb{N}^d}^{4(1-2\theta)}}\\
&\le C\,\Delta\tau^{2\theta}\sum_{\alpha\in\mathbb{N}^d}\tfrac{1}{|\alpha|_{\mathbb{N}^d}^{d+8\,\epsilon}}\\
&\le C\,\epsilon^{-1}\,\Delta\tau^{2(\frac{4-d}{8}-\epsilon)}.
\end{aligned}\tag{4.9}$$

`Estimation of` $\mathcal{B}_B^m$: Using, again, the definition of the Hilbert-Schmidt norm we have

$$\mathcal{B}_B^m = 2\sum_{\alpha\in\mathbb{N}^d}\left(\sum_{\ell=1}^{m}\int_{\Delta_\ell}\|\mathcal{S}(\tau_m-\tau_{\ell-1})\varepsilon_\alpha-\mathcal{S}(\tau_m-\tau)\varepsilon_\alpha\|_{0,D}^2\,d\tau\right).\tag{4.10}$$

Since $\mathcal{S}(t)\varepsilon_\alpha=e^{-\lambda_\alpha^2 t}\,\varepsilon_\alpha$ for $t\ge 0$, (4.10) yields

$$\begin{aligned}
\mathcal{B}_B^m &= 2\sum_{\alpha\in\mathbb{N}^d}\left[\sum_{\ell=1}^{m}\int_{\Delta_\ell}\left(\int_D\left[e^{-\lambda_\alpha^2(\tau_m-\tau_{\ell-1})}-e^{-\lambda_\alpha^2(\tau_m-\tau)}\right]^2\varepsilon_\alpha^2(x)\,dx\right)d\tau\right]\\
&= 2\sum_{\alpha\in\mathbb{N}^d}\left[\sum_{\ell=1}^{m}\int_{\Delta_\ell}e^{-2\lambda_\alpha^2(\tau_m-\tau)}\left[1-e^{-\lambda_\alpha^2(\tau-\tau_{\ell-1})}\right]^2d\tau\right]\\
&\le 2\sum_{\alpha\in\mathbb{N}^d}\left(1-e^{-\lambda_\alpha^2\,\Delta\tau}\right)^2\left[\int_0^{\tau_m}e^{-2\lambda_\alpha^2(\tau_m-\tau)}\,d\tau\right]\\
&\le \sum_{\alpha\in\mathbb{N}^d}\tfrac{1-e^{-2\lambda_\alpha^2\,\Delta\tau}}{\lambda_\alpha^2},
\end{aligned}$$

from which, applying (2.11), we obtain

$$\mathcal{B}_B^m \le C\,p_d(\Delta\tau^{\frac{1}{4}})\,\Delta\tau^{\frac{4-d}{4}}.\tag{4.11}$$

Thus, we obtain the estimate (4.6) as a conclusion of (4.8), (4.9) and (4.11). □

## 5. Convergence of the fully-discrete approximations

In this section, our goal is to derive a discrete in time $L_t^\infty(L_P^2(L_x^2))$ error estimate for the Backward Euler fully-discrete approximations of $\widehat{u}$ given in (1.7)–(1.8). For that, we follow the way to compare them to the Backward Euler time-discrete approximations of $\widehat{u}$ defined in (4.1)–(4.2), under the light of the error estimate obtained in Theorem 7.

Our first step, is to derive a discrete in time $L_t^2(L_x^2)$ error estimate between the Backward Euler time-discrete and the Backward Euler fully discrete approximations of the solution $w$ of (1.3) given below: Set

$$W_h^0 := P_h w_0, \tag{5.1}$$

and then, for $m = 1, \dots, M$, find $W_h^m \in M_h$ such that

$$W_h^m - W_h^{m-1} + \Delta\tau\, B_h W_h^m = 0. \tag{5.2}$$

**Proposition 8.** *Let $r \in \{2, 3, 4\}$, $w$ be the solution of the problem (1.3), $(W^m)_{m=0}^M$ be the Backward Euler time-discrete approximations of $w$ defined in (4.3)-(4.4), and $(W_h^m)_{m=0}^M$ be the Backward Euler fully-discrete approximations of $w$ specified in (5.1)-(5.2). If $w_0 \in \dot{\mathbf{H}}^3(D)$, then, there exists a constant $C > 0$, independent of $T$, $h$ and $\Delta\tau$, such that*

$$\left( \sum_{m=1}^M \Delta\tau \, \|W^m - W_h^m\|_{0,D}^2 \right)^{\frac{1}{2}} \le C\ h^{\nu(r,\theta)}\ \|w_0\|_{\dot{\mathbf{H}}^{\xi(r,\theta)}} \quad \forall\, \theta \in [0, 1], \tag{5.3}$$

*where*

$$\nu(r,\theta) := \begin{cases} 2\,\theta & \text{if } \ r = 2 \\ 4\,\theta & \text{if } \ r = 3 \\ 5\,\theta & \text{if } \ r = 4 \end{cases} \quad \text{and} \quad \xi(r,\theta) := \begin{cases} 3\theta - 2 & \text{if } \ r = 2 \\ 4\theta - 2 & \text{if } \ r = 3 \\ 5\theta - 2 & \text{if } \ r = 4 \end{cases}. \tag{5.4}$$

*Proof.* Let $E^m := W^m - W_h^m$ for $m = 0, \dots, M$. We will get (5.3) by interpolation, showing it for $\theta = 0$ and $\theta = 1$.

We use (4.4) and (5.2), to obtain: $T_{B,h}(E^m - E^{m-1}) + \Delta\tau\, E^m = \Delta\tau\,(T_B - T_{B,h})\Delta^2 W^m$ for $m = 1, \dots, M$. Taking the $L^2(D)-$inner product of both sides of the latter equation by $E^m$ and using (2.21), we arrive at

$$\begin{aligned} \|\Delta(T_{B,h}E^m)\|_{0,D}^2 - &(\Delta(T_{B,h}E^{m-1}), \Delta(T_{B,h}E^m))_{0,D} \\ &+ \Delta\tau\, \|E^m\|_{0,D}^2 = \Delta\tau\, ((T_B - T_{B,h})\Delta^2 W^m, E^m)_{0,D} \end{aligned} \tag{5.5}$$

for $m = 1, \dots, M$. Now, using the Cauchy-Schwartz inequality and the geometric mean inequality we obtain

$$-\,2\,(\Delta(T_{B,h}E^{m-1}), \Delta(T_{B,h}E^m))_{0,D} \ge -\left( \|\Delta(T_{B,h}E^{m-1})\|_{0,D}^2 + \|\Delta(T_{B,h}E^m)\|_{0,D}^2 \right) \tag{5.6}$$

for $m = 1, \dots, M$. Next, we combine (5.5) and (5.6) to conclude

$$\|\Delta(T_{B,h}E^m)\|_{0,D}^2 - \|\Delta(T_{B,h}E^{m-1})\|_{0,D}^2 + 2\,\Delta\tau\, \|E^m\|_{0,D}^2 \le 2\,\Delta\tau\, ((T_B - T_{B,h})\Delta^2 W^m, E^m)_{0,D}$$

for $m = 1, \dots, M$. Summing with respect to $m$ from 1 up to $M$, applying the Cauchy-Schwarz inequality and using that $T_{B,h}E^0 = 0$, we obtain

$$\sum_{m=1}^M \Delta\tau\, \|E^m\|_{0,D}^2 \le \sum_{m=1}^M \Delta\tau\, \left\|(T_B - T_{B,h})\Delta^2 W^m\right\|_{0,D}^2. \tag{5.7}$$

Let $r = 3$. Then, by (2.24) and (5.7), we obtain

$$\left( \sum_{m=1}^M \Delta\tau\, \|E^m\|_{0,D}^2 \right)^{\frac{1}{2}} \le C\, h^4 \left( \sum_{m=1}^M \Delta\tau\, \left\|\Delta^2 W^m\right\|_{0,D}^2 \right)^{\frac{1}{2}}. \tag{5.8}$$

Taking the $(\cdot,\cdot)_{0,D}-$inner product of (4.4) with $\Delta^2 W^m$, and then integrating by parts and summing with respect to $m$ from 1 up to $M$, it follows that

$$\sum_{m=1}^{M}(\Delta W^m-\Delta W^{m-1},\Delta W^m)_{0,D}+\sum_{m=1}^{M}\Delta\tau\,\|\Delta^2 W^m\|_{0,D}^2=0. \tag{5.9}$$

Since $\sum_{m=1}^{M}\left(\Delta W^m-\Delta W^{m-1},\Delta W^m\right)_{0,D}\geq\frac{1}{2}\left(\|\Delta W^M\|_{0,D}^2-\|\Delta W^0\|_{0,D}^2\right)$, (5.9) yields

$$\sum_{m=1}^{M}\Delta\tau\,\|\Delta^2 W^m\|_{0,D}^2\leq\tfrac{1}{2}\,\|w_0\|_{2,D}^2. \tag{5.10}$$

Combining, now, (5.8), (5.10) and (2.3), we obtain

$$\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|_{0,D}^2\right)^{\frac{1}{2}}\leq C\,h^4\,\|w_0\|_{\dot{\mathbf{H}}^2}. \tag{5.11}$$

Let $r=2$. Then, by (2.24), (2.4) and (5.7), we obtain

$$\begin{aligned}\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|_{0,D}^2\right)^{\frac{1}{2}}&\leq C\,h^2\,\left(\sum_{m=1}^{M}\Delta\tau\,\left\|\Delta^2 W^m\right\|_{\dot{\mathbf{H}}^{-1}}^2\right)^{\frac{1}{2}}\\&\leq C\,h^2\,\left[-\sum_{m=1}^{M}\Delta\tau\,(T_E\Delta^2 W^m,\Delta^2 W^m)_{0,D}\right]^{\frac{1}{2}}\\&\leq C\,h^2\,\left[-\sum_{m=1}^{M}\Delta\tau\,(\Delta W^m,\Delta^2 W^m)_{0,D}\right]^{\frac{1}{2}}.\end{aligned} \tag{5.12}$$

Taking the $(\cdot,\cdot)_{0,D}-$inner product of (4.4) with $\Delta W^m$, integrating by parts and summing with respect to $m$ from 1 up to $M$, it follows that

$$\sum_{m=1}^{M}\left(\nabla W^m-\nabla W^{m-1},\nabla W^m\right)_{0,D}-\sum_{m=1}^{M}\Delta\tau\,(\Delta^2 W^m,\Delta W^m)_{0,D}=0. \tag{5.13}$$

Since $\sum_{m=1}^{M}(\nabla W^m-\nabla W^{m-1},\nabla W^m)_{0,D}\geq\frac{1}{2}\left[\|\nabla W^M\|_{0,D}^2-\|\nabla W^0\|_{0,D}^2\right]$, (5.13) yields

$$-\sum_{m=1}^{M}\Delta\tau\,(\Delta^2 W^m,\Delta W^m)_{0,D}\leq\tfrac{1}{2}\,\|w_0\|_{1,D}^2. \tag{5.14}$$

Combining (5.12), (5.14) and (2.3) we get

$$\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|_{0,D}^2\right)^{\frac{1}{2}}\leq C\,h^2\,\|w_0\|_{\dot{\mathbf{H}}^1}. \tag{5.15}$$

Let $r=4$. Then, observing that $\Delta^2 W^m\in\dot{\mathbf{H}}^2(D)$ and using the relations (2.24), (2.4) and (5.7), we obtain

$$
\begin{aligned}
\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|_{0,D}^2\right)^{\frac{1}{2}} &\leq C\,h^5\,\left(\sum_{m=1}^{M}\Delta\tau\,\left\|\Delta^2W^m\right\|_{\dot{\mathbf{H}}^1}^2\right)^{\frac{1}{2}}\\
&\leq C\,h^5\,\left(\sum_{m=1}^{M}\Delta\tau\,\left\|\Delta^3W^m\right\|_{\dot{\mathbf{H}}^{-1}}^2\right)^{\frac{1}{2}}\\
&\leq C\,h^5\,\left[-\sum_{m=1}^{M}\Delta\tau\,(T_E\Delta^3W^m,\Delta^3W^m)_{0,D}\right]^{\frac{1}{2}}\\
&\leq C\,h^5\,\left[-\sum_{m=1}^{M}\Delta\tau\,(\Delta^2W^m,\Delta^3W^m)_{0,D}\right]^{\frac{1}{2}}.
\end{aligned}
\tag{5.16}
$$

After, applying the operator $\Delta$ on (4.4), take the $(\cdot,\cdot)_{0,D}-$inner product of the obtained relation with $\Delta^2W^m$, integrate by parts and sum with respect to $m$ from 1 up to $M$, to get

$$
-\sum_{m=1}^{M}\left(\Delta W^m-\Delta W^{m-1},\Delta^2W^m\right)_{0,D}-\sum_{m=1}^{M}\Delta\tau\,(\Delta^3W^m,\Delta^2W^m)_{0,D}=0. \tag{5.17}
$$

Also, we have

$$
\begin{aligned}
-\sum_{m=1}^{M}(\Delta W^m-\Delta W^{m-1},\Delta^2W^m)_{0,D} &\geq \sum_{m=1}^{M}\left(\,\|\Delta W^m\|_{\dot{\mathbf{H}}^1}^2-\|\Delta W^m\|_{\dot{\mathbf{H}}^1}\,\|\Delta W^{m-1}\|_{\dot{\mathbf{H}}^1}\,\right)\\
&\geq \tfrac{1}{2}\,\sum_{m=1}^{M}\left(\,\|\Delta W^m\|_{\dot{\mathbf{H}}^1}^2-\|\Delta W^{m-1}\|_{\dot{\mathbf{H}}^1}^2\,\right)\\
&\geq \tfrac{1}{2}\,\left(\,\|\Delta W^M\|_{\dot{\mathbf{H}}^1}^2-\|\Delta W^0\|_{\dot{\mathbf{H}}^1}\,\right).
\end{aligned}
\tag{5.18}
$$

Thus, (5.17) and (5.18) yield

$$
-\sum_{m=1}^{M}\Delta\tau\,(\Delta^3W^m,\Delta^2W^m)_{0,D}\leq\;\tfrac{1}{2}\,\|w_0\|_{\dot{\mathbf{H}}^3}^2. \tag{5.19}
$$

Combining (5.16) and (5.19) we get

$$
\left(\sum_{m=1}^{M}\Delta\tau\,\|E^m\|_{0,D}^2\right)^{\frac{1}{2}}\;\leq\;C\,h^5\,\|w_0\|_{\dot{\mathbf{H}}^3}. \tag{5.20}
$$

Thus, the relations (5.11), (5.15) and (5.20) yield (5.3) for $\theta=1$.

Since $T_{B,h}(W_h^m-W_h^{m-1})+\Delta\tau\,W_h^m=0$ for $m=1,\dots,M$, we obtain

$$
\tfrac{1}{2}\sum_{m=1}^{M}\left[\,\|\Delta(T_{B,h}W_h^m)\|_{0,D}^2-\|\Delta(T_{B,h}W_h^{m-1})\|_{0,D}^2\,\right]+\sum_{m=1}^{M}\Delta\tau\,\|W_h^m\|_{0,D}^2\leq 0,
$$

which, along with (2.22) and (2.4), yields

$$
\begin{aligned}
\left(\sum_{m=1}^{M}\Delta\tau\,\|W_h^m\|_{0,D}^2\right)^{\frac{1}{2}} &\leq \tfrac{1}{\sqrt{2}}\,\|\Delta(T_{B,h}w_0)\|_{0,D}\\
&\leq C\,\|w_0\|_{\dot{\mathbf{H}}^{-2}}.
\end{aligned}
\tag{5.21}
$$

Now, using (4.4) and (2.17), we obtain $(T_E W^m - T_E W^{m-1}, T_E W^m)_{0,D} + \Delta\tau\, \|W^m\|_{0,D}^2 = 0$ for $m = 1, \dots, M$, which yields $\|T_E W^m\|_{0,D}^2 - \|T_E W^{m-1}\|_{0,D}^2 + 2\,\Delta\tau\, \|W^m\|_{0,D}^2 \leq 0$ for $m = 1, \dots, M$. Then, summing with respect to $m$ from 1 up to $M$, and using (2.13) and (2.4) we obtain

$$\begin{aligned}\left(\sum_{k=1}^{M} \Delta\tau\, \|W^m\|_{0,D}^2\right)^{\frac{1}{2}} &\leq \tfrac{1}{\sqrt{2}}\, \|T_E w^0\|_{0,D} \\ &\leq C\, \|w_0\|_{-2,D} \\ &\leq C\, \|w_0\|_{\dot{\mathbf{H}}^{-2}}.\end{aligned} \tag{5.22}$$

Finally, combine (5.21) with (5.22) to get $\left(\sum_{m=1}^{M} \Delta\tau\, \|E^m\|_{0,D}^2\right)^{\frac{1}{2}} \leq C\, \|w_0\|_{\dot{\mathbf{H}}^{-2}}$, which is equivalent to (5.3) for $\theta = 0$. □

The following lemma ensures the existence of a continuous Green function for the solution operator of a discrete elliptic problem.

**Lemma 9.** *Let $r \in \{2, 3, 4\}$, $\epsilon > 0$, $f \in L^2(D)$ and $\psi_h \in M_h$ such that*

$$\epsilon\, B_h \psi_h + \psi_h = P_h f. \tag{5.23}$$

*Then there exists a function $G_{h,\epsilon} \in C(\overline{D \times D})$ such that*

$$\psi_h(x) = \int_D G_{h,\epsilon}(x, y)\, f(y)\, dy \quad \forall\, x \in \overline{D} \tag{5.24}$$

*and $G_{h,\epsilon}(x, y) = G_{h,\epsilon}(y, x)$ for $x, y \in \overline{D}$.*

*Proof.* Let $\dim(M_h) = n_h$ and $\gamma_h : M_h \times M_h \to \mathbb{R}$ be an inner product on $M_h$ given by $\gamma_h(\chi_A, \chi_B) := (\Delta\chi_A, \Delta\chi_B)_{0,D}$ for $\chi_A$, $\chi_B \in M_h$. We can construct a basis $(\chi_j)_{j=1}^{n_h}$ of $M_h$ which is $L^2(D)-$orthonormal, i.e., $(\chi_i, \chi_j)_{0,D} = \delta_{ij}$ for $i, j = 1, \dots, n_h$, and $\gamma_h-$orthogonal, i.e., there are $(\lambda_{h,\ell})_{\ell=1}^{n_h} \subset (0, +\infty)$ such that $\gamma_h(\chi_i, \chi_j) = \lambda_{h,i}\, \delta_{ij}$ for $i, j = 1, \dots, n_h$ (see Section 8.7 in [9]). Thus, there are $(\mu_j)_{j=1}^{n_h} \subset \mathbb{R}$ such that $\psi_h = \sum_{j=1}^{n_h} \mu_j\, \chi_j$, and (5.23) is equivalent to $\mu_i = \frac{1}{1+\epsilon\, \lambda_{h,i}}\, (f, \chi_i)_{0,D}$ for $i = 1, \dots, n_h$. Finally, we obtain (5.24) with $G_{h,\epsilon}(x, y) = \sum_{j=1}^{n_h} \frac{\chi_j(x)\chi_j(y)}{1+\epsilon\, \lambda_{h,j}}$. □

We are ready to compare, in the discrete in time $L_t^\infty(L_P^2(L_x^2))$ norm, the time-discrete with the fully-discrete Backward Euler approximations of $\widehat{u}$.

**Proposition 10.** *Let $r \in \{2, 3, 4\}$, $\widehat{u}$ be the solution of the problem* (1.5), *$(\widehat{U}_h^m)_{m=0}^{M}$ be the Backward Euler fully-discrete approximations of $\widehat{u}$ specified in* (1.7)-(1.8), *and $(\widehat{U}^m)_{m=0}^{M}$ be the Backward Euler time-discrete approximations of $\widehat{u}$ specified in* (4.1)-(4.2). *Then, there exists a constant $C > 0$, independent of $\Delta x$, $\Delta t$, $h$ and $\Delta\tau$, such that*

$$\max_{1 \leq m \leq M} \left\{ \mathbb{E}\left[ \left\| \widehat{U}_h^m - \widehat{U}^m \right\|_{0,D}^2 \right] \right\}^{\frac{1}{2}} \leq C\, \epsilon^{-\frac{1}{2}}\ \ h^{\nu_\star(r,d)-\epsilon}, \quad \forall\, \epsilon \in (0, \nu_\star(r, d)] \tag{5.25}$$

*where*

$$\nu_\star(r, d) := \begin{cases} \frac{4-d}{3} & \text{if} \quad r = 2 \\ \frac{4-d}{2} & \text{if} \quad r = 3, 4 \end{cases}. \tag{5.26}$$

*Proof.* Let $I : L^2(D) \to L^2(D)$ be the identity operator and $\Lambda_h : L^2(D) \to S_h^r$ be the inverse discrete elliptic operator given by $\Lambda_h := (I + \Delta\tau\, B_h)^{-1} P_h$ and having a Green function $G_{h,\Delta\tau}$ (cf. Lemma 9). Also, for $\ell \in \mathbb{N}$, we denote by $G_{h,\Delta\tau,\ell}$ the Green function of $\Lambda_h^\ell$. Using, now, an induction argument, from (1.8) we conclude that $\widehat{U}_h^m = \sum_{j=1}^{m} \int_{\Delta_j} \Lambda_h^{m-j+1} \widehat{W}(\tau, \cdot)\, d\tau$, $m = 1, \dots, M$, which is written, equivalently, as follows:

$$\widehat{U}_h^m(x) = \int_0^{\tau_m} \int_D \widehat{\mathcal{D}}_{h,m}(\tau; x, y)\, \widehat{W}(\tau, y)\, dy d\tau \quad \forall\, x \in \overline{D}, \quad m = 1, \dots, M, \tag{5.27}$$

where

$$\widehat{\mathcal{D}}_{h,m}(\tau;x,y) := \sum_{j=1}^{m} \mathcal{X}_{\Delta_j}(\tau)\, G_{h,\Delta\tau,m-j+1}(x,y) \quad \forall\,\tau\in[0,T], \quad \forall\,x,y\in D.$$

Using (4.7), (5.27), the Itô-isometry property of the stochastic integral (2.6), (2.5) and (2.8), we get

$$\begin{aligned}\mathbb{E}\left[\|\widehat{U}^m-\widehat{U}_h^m\|_{0,D}^2\right] &\le \int_0^{\tau_m}\Big(\int_D\int_D\big[\widehat{\mathcal{K}}_m(\tau;x,y)-\widehat{\mathcal{D}}_{h,m}(\tau;x,y)\big]^2\,dydx\Big)\,d\tau\\ &\le \sum_{j=1}^{m}\int_{\Delta_j}\|\Lambda^{m-j+1}-\Lambda_h^{m-j+1}\|_{\mathrm{HS}}^2\,d\tau, \quad m=1,\dots,M,\end{aligned}$$

where $\Lambda$ is the inverse elliptic operator defined in the proof of Theorem 7. Now, we use the definition of the Hilbert-Schmidt norm and the deterministic error estimate (5.3), to have

$$\begin{aligned}\mathbb{E}\left[\|\widehat{U}^m-\widehat{U}_h^m\|_{0,D}^2\right] &\le \sum_{j=1}^{m}\Delta\tau\left[\sum_{\alpha\in\mathbb{N}^d}\|\Lambda^{m-j+1}\varepsilon_\alpha-\Lambda_h^{m-j+1}\varepsilon_\alpha\|_{0,D}^2\right]\\ &\le \sum_{\alpha\in\mathbb{N}^d}\left[\sum_{j=1}^{m}\Delta\tau\,\|\Lambda^j\varepsilon_\alpha-\Lambda_h^j\varepsilon_\alpha\|_{0,D}^2\right]\\ &\le C\,h^{2\,\nu(r,\theta)}\sum_{\alpha\in\mathbb{N}^d}\|\varepsilon_\alpha\|_{\dot{\mathbf{H}}^{\xi(r,\theta)}}^2, \quad m=1,\dots,M, \quad \forall\,\theta\in[0,1].\end{aligned}$$

Thus, we arrive at

$$\max_{1\le m\le M}\Big(\mathbb{E}\left[\|\widehat{U}^m-\widehat{U}_h^m\|_{0,D}^2\right]\Big)^{\frac12} \le C\,h^{\nu(r,\theta)}\left(\sum_{\alpha\in\mathbb{N}^d}|\alpha|_{\mathbb{N}^d}^{2\,\xi(r,\theta)}\right)^{\frac12}, \quad \forall\,\theta\in[0,1],$$

from which, requiring $-2\,\xi(r,\theta)>d$ and using (2.10), (5.25), easily, follows. □

The available error estimates allow us to conclude a discrete in time $L_t^\infty(L_P^2(L_x^2))$ convergence of the Backward Euler fully-discrete approximations of $\widehat{u}$, over a uniform partition of $[0,T]$.

**Theorem 11.** *Let $r\in\{2,3,4\}$, $\nu_\star(r,d)$ be defined by (5.26), $\widehat{u}$ be the solution of problem (1.5), and $(\widehat{U}_h^m)_{m=0}^M$ be the Backward Euler fully-discrete approximations of $\widehat{u}$ constructed by (1.7)-(1.8). Then, there exists a constant $C>0$, independent of $T$, $h$, $\Delta\tau$, $\Delta t$ and $\Delta x$, such that*

$$\max_{0\le m\le M}\left\{\mathbb{E}\left[\|\widehat{U}_h^m-\widehat{u}(\tau_m,\cdot)\|_{0,D}^2\right]\right\}^{\frac12} \le C\,\left[\widetilde{\omega}(\Delta\tau,\epsilon_1)\,\Delta\tau^{\frac{4-d}{8}-\epsilon_1}+\epsilon_2^{-\frac12}\,h^{\nu_\star(r,d)-\epsilon_2}\right], \tag{5.28}$$

*for $\epsilon_1\in\big(0,\frac{4-d}{8}\big]$ and $\epsilon_2\in\big(0,\nu_\star(r,d)\big]$ where $\widetilde{\omega}(\Delta\tau,\epsilon_1):=\epsilon_1^{-\frac12}+(\Delta\tau)^{\epsilon_1}(p_d(\Delta\tau^{\frac14}))^{\frac12}$.*

*Proof.* The estimate is a simple consequence of the error bounds (5.25) and (4.6). □

**Remark 3.** Let us find the optimal value for the parameters $\epsilon_1$ and $\epsilon_2$ in (5.28) and for parameter $\epsilon$ in (3.1). Let $g(\epsilon)=\epsilon^{-\frac12}\,\delta^{-\epsilon}$ for $\epsilon\in(0,\gamma]$ where $\gamma$, $\delta\in(0,1)$. Then, a simple calculation yields

$$g'(\epsilon)=\epsilon^{-\frac32}\,\delta^{-\epsilon}\,(\,\epsilon-\widetilde{\epsilon}(\delta)\,)\,(\,\epsilon+\widetilde{\epsilon}(\delta)\,), \quad \forall\,\epsilon\in(0,\gamma],$$

where $\widetilde{\epsilon}(\delta):=2^{-\frac12}\,|\log(\delta)|^{-\frac12}$. Since $\lim_{\delta\to0}\widetilde{\epsilon}(\delta)=0$, there exists $\delta_\gamma\in(0,1)$ such that $\widetilde{\epsilon}(\delta)\in(0,\gamma)$ for $\delta\in(0,\delta_\gamma]$. Now, assuming that $\delta\in(0,\delta_\gamma]$, we conclude that

$$\min_{\epsilon\in(0,\gamma]}g(\epsilon)=g\left(\widetilde{\epsilon}(\delta)\right)=2^{\frac14}\,|\log(\delta)|^{\frac14}\,\delta^{-\frac{1}{\sqrt{2}\,\sqrt{|\log(\delta)|}}}.$$

Thus, assuming that $h$ and $\Delta\tau$ are small enough, and setting $\epsilon_1 = \widetilde{\epsilon}(\Delta\tau)$ and $\epsilon_2 = \widetilde{\epsilon}(h)$, the error estimate (5.28) is written in the form

$$O\left(\Delta\tau^{\frac{4-d}{8}-\frac{1}{\sqrt{2}\sqrt{|\log(\Delta\tau)|}}}\,|\log(\Delta\tau)|^{\frac{1}{4}} + h^{\nu_\star(r,d)-\frac{1}{\sqrt{2}\sqrt{|\log(h)|}}}\,|\log(h)|^{\frac{1}{4}}\right).$$

Proceeding in a similar way, the error bound (3.1) is written as

$$O\left(\Delta t^{\frac{4-d}{8}} + \Delta x^{\frac{4-d}{2}-\frac{1}{\sqrt{2}\sqrt{|\log(\Delta x)|}}}\,|\log(\Delta x)|^{\frac{1}{4}}\right).$$

**Remark 4.** The solution $u$ of (1.1) is $\beta-$Hölder in $t$ and $\beta'-$Hölder in $x$ with $\beta < \frac{4-d}{8}$ and $\beta' < \frac{4-d}{2}$ (see, e.g., [5], [10]). This is the reason why the expected order of convergence in time and space, are respectively $\beta$ and $\beta'$. According to Theorem 11, the expected order of convergence in time is achieved and the expected order of convergence in space is also achieved when $r = 3, 4$. For $r = 2$, the order of convergence in space is lower and an explanation for that is the fact that the order of convergence in the $L^2(D)-$norm of the finite element method for the biharmonic problem is equal to 2 and not equal to $r+1 = 3$ as it is for $r = 3, 4$ (see Proposition 4). The expected order of convergence in time and in space are also obtained in [4] and [21] for other type of numerical methods.

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